

Then, Sakata (1956) proposed, as a natural extension of the Fermi-Yang model, that in order to incorporate strangeness, the proton, the neutron, and  $\Lambda$  hyperon are to be considered as fundamental particles.  $\Sigma$  and  $\Xi$  must be built as composite baryons made of two of  $(p, n, \Lambda)$  and one of  $(\bar{p}, \bar{n}, \bar{\Lambda})$  as  $\Sigma, \Xi = \Lambda(\bar{N}N)_{I=1}$  and  $\Lambda\bar{\Lambda}\bar{N}$ . The model predicted correctly eight (octet) mesons, but when they tested the prediction of the anomalous magnetic moments of hyperons, the model failed. Gell-Mann (1963) replaced  $(p, n, \Lambda)$  by the hypothetical fermions named quarks,  $(u, d, s)$ . We start with Gell-Mann's quark model.

## 2.2 SU(3) symmetry

### 2.2a Group theory of SU(3)

To discuss mathematics of SU(3), we will not treat quark fields as quantized (anti-commuting) fields, but change their ordering freely. Suppress their spin structure for the time being, since it is inessential to group theory of SU(3).

SU(3) = all unitary transformations on three-component complex vectors less the overall common phase rotation (called U(1), an abelian group).

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow \begin{pmatrix} u' \\ d' \\ s' \end{pmatrix} = U \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad \text{with } U^\dagger U = UU^\dagger = 1 \text{ and } \det U = 1.$$

The condition  $\det U = 1$  removes the common phase transformation:  $U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} e^{i\delta}$ .

According to a general theorem, any unitary transformation can be written in terms of exponentiated hermitian operators as  $U = \exp(iH)$ . In the present case, if we exhaust all  $3 \times 3$  hermitian matrices for  $H$ , we include all possible  $3 \times 3$  unitary transformations;

$$U = \exp\left[i \sum_{a=1}^8 \frac{1}{2} \lambda_a \alpha_a\right]$$

with

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix},$$

We do not include a hermitian matrix  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ , since it would contradict with  $\det U = 1$ .

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The SU(2) isospin rotation is a subgroup consisting of  $\frac{1}{2}(\lambda_1, \lambda_2, \lambda_3)$ . There are two more SU(2) subgroups;

$$\left( \frac{1}{2}\lambda_4, \frac{1}{2}\lambda_5, \frac{1}{4}\lambda_3 + \frac{\sqrt{3}}{4}\lambda_8 \right) \quad \text{and} \quad \left( \frac{1}{2}\lambda_6, \frac{1}{2}\lambda_7, -\frac{1}{4}\lambda_3 + \frac{\sqrt{3}}{4}\lambda_8 \right).$$

They are sometimes called V-spin and U-spin subgroups, respectively.

The eight matrices  $\lambda_a$  ( $a=1, 2, \dots, 8$ ) are  $3 \times 3$  matrix realization (representation) of the algebra of SU(3):

$$\left[ \frac{1}{2}\lambda_a, \frac{1}{2}\lambda_b \right] = i f_{abc} \frac{1}{2}\lambda_c \quad (c \text{ summed over from 1 to 8}).$$

The three-dimensional column vector consisting of  $u, d, s$  is (the vector space of) the 3-dimensional representation of SU(3) group. It is possible to realize the SU(3) algebra

$$[\Lambda_a, \Lambda_b] = i f_{abc} \Lambda_c$$

with 8 matrices of  $(N \times N)$ . Such representation is called an  $N$ -dimensional representation of  $SU(3)$  and  $N$  is not arbitrary (unlike  $SU(2)$ ). The coefficient  $f_{abc}$  is called the structure constant of  $SU(3)$  that determines completely the group structure of  $SU(3)$ . It possesses the property  $f_{abc} = -f_{acb} = -f_{cba} = -f_{bac}$  (totally antisymmetric under permutation of a pair of indices) and its numerical values (with the choice of  $\lambda_a$  as given in the previous page) are:

$$\begin{aligned} f_{123} = 1, \quad f_{147} = -f_{156} = f_{246} = f_{257} = f_{345} = -f_{367} = 1/2, \\ f_{458} = f_{678} = \sqrt{3}/2. \quad \text{Other } f_{abc} \text{ not related to these are all } 0. \end{aligned}$$

When one takes anti-commutation relations of  $\lambda_a$ , one finds

$$\left\{ \frac{1}{2} \lambda_a, \frac{1}{2} \lambda_b \right\} = d_{abc} \frac{1}{2} \lambda_c.$$

Here, for convenience, we include  $\lambda_0 = \sqrt{\frac{2}{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  to close the anti-commutation relations.

$\lambda_0$  is not one of the  $SU(3)$  generators (the name for  $\lambda_a$  ( $a=1,2,\dots,8$ )). The coefficient  $d_{abc}$  is given by ( $a,b,c = 0,1,2,\dots,8$ )

$$\begin{aligned} d_{abc} = d_{acb} = d_{cba} = d_{bac} \quad (\text{totally symmetric}) \\ d_{118} = d_{228} = d_{338} = \sqrt{1/3}, \quad d_{448} = d_{558} = d_{668} = d_{778} = -\frac{1}{2\sqrt{3}}, \quad d_{888} = -\sqrt{\frac{1}{3}}, \\ d_{146} = d_{157} = d_{247} = d_{256} = d_{344} = d_{355} = -d_{366} = -d_{377} = \frac{1}{2}, \\ d_{ab0} = \sqrt{\frac{2}{3}} \delta_{ab} \quad (i,j = 0,1,2,\dots,8). \quad \text{Other } d_{abc} \text{ not related to these are all zero.} \end{aligned}$$

It should be noted that the anti-commutation relations are not the common relations of the  $SU(3)$  group; for general  $N \times N$  representation,  $\{\lambda_a, \lambda_b\} \neq d_{abc} \lambda_c$ .

The following is a comparison chart between  $SU(2)$  and  $SU(3)$ .

$$\begin{aligned} \begin{pmatrix} p \\ n \end{pmatrix} \rightarrow \exp\left(\frac{i}{2} \sum_{a=1}^3 \tau_a \cdot \alpha_a\right) \begin{pmatrix} p \\ n \end{pmatrix}, \quad \Leftrightarrow \quad \begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow \exp\left(\frac{i}{2} \sum_{a=1}^8 \lambda_a \alpha_a\right) \begin{pmatrix} u \\ d \\ s \end{pmatrix} \\ \left[\frac{1}{2} \tau_a, \frac{1}{2} \tau_b\right] = i \varepsilon_{abc} \frac{1}{2} \tau_c, \quad \Leftrightarrow \quad \left[\frac{1}{2} \lambda_a, \frac{1}{2} \lambda_b\right] = i f_{abc} \frac{1}{2} \lambda_c. \\ \left\{ \frac{1}{2} \tau_a, \frac{1}{2} \tau_b \right\} = \delta_{ab} \frac{1}{2} \mathbb{1}, \quad \Leftrightarrow \quad \left\{ \frac{1}{2} \lambda_a, \frac{1}{2} \lambda_b \right\} = d_{abc} \frac{1}{2} \lambda_c. \end{aligned}$$

### Representations of $SU(3)$

#### (a) Fundamental (spinor) representation.

The 3-dimensional (column vector) representation is called the fundamental representation of  $SU(3)$  in the sense that all other representations are constructed from products of this 3-dimensional representation (even  $\bar{3}$  representation).

$$\begin{pmatrix} u \\ d \\ s \end{pmatrix} \rightarrow \exp\left(\frac{i}{2} \lambda_a \alpha_a\right) \begin{pmatrix} u \\ d \\ s \end{pmatrix} \quad (3 \text{ dim. representation})$$

$$(\bar{u}, \bar{d}, \bar{s}) = (\bar{u}, \bar{d}, \bar{s}) \exp\left(-\frac{i}{2} \lambda_a \alpha_a\right) \quad (\bar{3} \text{ dim. representation}).$$

The scalar product  $(\bar{u}, \bar{d}, \bar{s}) \cdot \begin{pmatrix} u \\ d \\ s \end{pmatrix}$  is obviously invariant under  $SU(3)$  transformations.

Though the dimensions are the same, 3 and  $\bar{3}$  are inequivalent unlike 2 and  $\bar{2}$  in  $SU(2)$ .

Theorem:  $\bar{q}^i$  transforms like  $\epsilon^{ijk} q_j(1) q_k(2)$  under  $SU(3)$  where  $\epsilon^{ijk}$  is the totally antisymmetric tensors. [The number 1,2 in the parentheses refer to coordinates or momenta.]

Proof: We first show that  $\epsilon^{ijk} q_j(2) q_k(3) q_i(1)$  is invariant under  $SU(3)$ .

Under  $SU(3)$  rotations,

$$\begin{aligned} \det q &\equiv \begin{vmatrix} q_1(1) & q_1(2) & q_1(3) \\ q_2(1) & q_2(2) & q_2(3) \\ q_3(1) & q_3(2) & q_3(3) \end{vmatrix} \rightarrow \begin{vmatrix} U_1^i q_i(1) & U_1^i q_i(2) & U_1^i q_i(3) \\ U_2^i q_i(1) & U_2^i q_i(2) & U_2^i q_i(3) \\ U_3^i q_i(1) & U_3^i q_i(2) & U_3^i q_i(3) \end{vmatrix} \\ &= \begin{vmatrix} U_1^1 & U_1^2 & U_1^3 \\ U_2^1 & U_2^2 & U_2^3 \\ U_3^1 & U_3^2 & U_3^3 \end{vmatrix} \begin{vmatrix} q_1(1) & q_1(2) & q_1(3) \\ q_2(1) & q_2(2) & q_2(3) \\ q_3(1) & q_3(2) & q_3(3) \end{vmatrix} = \det U \cdot \det q = \det q. \end{aligned}$$

This means that

$$\begin{pmatrix} \epsilon^{1jk} & q_j(1) q_k(2) \\ \epsilon^{2jk} & q_j(1) q_k(2) \\ \epsilon^{3jk} & q_j(1) q_k(2) \end{pmatrix} \rightarrow \begin{pmatrix} \epsilon^{1jk} & q_j(1) q_k(2) \\ \epsilon^{2jk} & q_j(1) q_k(2) \\ \epsilon^{3jk} & q_j(1) q_k(2) \end{pmatrix} \exp\left(-\frac{i}{2} \lambda_a \alpha_a\right).$$

This theorem works for  $SU(N)$  group with the replacement  $\epsilon^{ijk} \dots \epsilon^{N-1}$   $q_j q_k \dots q_m$ .

In particular, for  $SU(2)$ ,  $\epsilon^{ij} q_j$  is equivalent to  $\bar{q}^i$ . [Note that  $q' = i \sigma_2 \bar{q}^t$ .]

(b) Products of  $\underline{3}$  or  $(\underline{3}$  and  $\bar{\underline{3}})$

Take a product of two  $\underline{3}$ 's. If the product is symmetrized or antisymmetrized in particle labels, they remain symmetric or antisymmetric even after  $SU(3)$  rotations: They make invariant subspaces.

$$\begin{aligned} q_i(1) \otimes q_j(2) &= \begin{pmatrix} q_i(1) q_j(2) + q_j(1) q_i(2) \\ uu, (ud+du)/\sqrt{2}, dd \\ (us+su)/\sqrt{2}, (ds+sd)/\sqrt{2} \end{pmatrix} + \begin{pmatrix} q_i(1) q_j(2) - q_j(1) q_i(2) \\ (ud-du)/\sqrt{2} \\ (su-us)/\sqrt{2}, (ds-sd)/\sqrt{2} \end{pmatrix} \\ &\quad \text{ss.} \\ &= \underline{6} + \bar{\underline{3}}. \end{aligned}$$

$$\begin{aligned} q_i(1) \otimes q_j(2) \otimes q_k(3) &= (\text{totally symmetric in } (123)) \\ &\quad + (\text{symmetrize in } (12), \text{ then antisymmetrize in } (13)) \\ &\quad + (\text{symmetrize in } (13), \text{ then antisymmetrize in } (12)) \\ &\quad + (\text{totally antisymmetrize in } (123)) \end{aligned}$$

$$\begin{aligned} \underline{10} \quad \boxed{\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}} &\quad uu, (uud+udu+duu)/\sqrt{3}, \quad (udd+dud+ddu)/\sqrt{3}, \quad ddd \\ &\quad (uus+usu+suu)/\sqrt{3}, \quad (uds+dus+usd+dsu+sud+sdu)/\sqrt{6}, \quad (dds+dsd+sdd)/\sqrt{3} \\ &\quad (uss+sus+ssu)/\sqrt{3}, \quad (dss+sds+ssd)/\sqrt{3} \\ &\quad \text{sss.} \end{aligned}$$

$$\begin{aligned} \underline{8} \quad \boxed{\begin{smallmatrix} 1 & 2 \\ 1 & 3 \end{smallmatrix}} &\quad q_i(1) q_j(2) q_k(3) + q_j(1) q_i(2) q_k(3) - q_k(1) q_j(2) q_i(3) - q_k(1) q_i(2) q_j(3) \\ &\quad \sim \epsilon_{ikn} \bar{q}^n(1,3) q_j(2) + \epsilon_{jkn} \bar{q}^n(1,3) q_i(2) \end{aligned}$$

$$\begin{aligned} \underline{8} \quad \boxed{\begin{smallmatrix} 1 & 3 \\ 2 & 1 \end{smallmatrix}} &\quad q_i(1) q_j(2) q_k(3) + q_k(1) q_j(2) q_i(3) - q_j(1) q_i(2) q_k(3) - q_j(1) q_k(2) q_i(3) \\ &\quad \sim \epsilon_{ijm} \bar{q}^m(1,2) q_k(3) + \epsilon_{kjm} \bar{q}^m(1,2) q_i(3) \end{aligned}$$

$$\frac{1}{\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}} \varepsilon^{ijk} q_i(1) q_j(2) q_k(3) .$$

The eight-dimensional subspace obtainable by symmetrizing in (23) and then antisymmetrizing in (12) is linearly dependent on the two 8 dimensional representations written above. Therefore,

$$\underline{3} \otimes \underline{3} \otimes \underline{3} = \underline{1} + \underline{8} + \underline{8} + \underline{10}.$$

The eight dimensional representation can be cast in the form of

$$\bar{q}^i(1) q_j(2) - \frac{1}{3} \delta_j^i \sum_k \bar{q}^k(1) q_k(2) , \text{ or } \bar{q}(1) \lambda_a q(2) / \sqrt{2} .$$

As you can see from the form  $\bar{q} \lambda_a q$ , they transform exactly like the generators of the group. Such representation is called the adjoint representation. Obviously, the dimension of the adjoint representation is the dimension of the generators of the group.

In order to construct irreducible representations from products of fundamental representation (3), the following theorem is useful.

#### Theorem (Weyl)

Irreducible representations of SU(N) groups are also irreducible representations of permutation group. The irreducible representations of permutation group are given by symmetrizing and then antisymmetrizing indices according to the Young tableaux. [This theorem works when you have only 3's or  $\bar{3}$ 's, not 3 and  $\bar{3}$  in coexistence.]

We will not prove the theorem here (see H. Weyl, "Classical Groups" or M. Hamermesh, Group Theory and its Applications to Physics Problems.)

It is easy to understand that tensors of definite permutation symmetry make an invariant subspace (though irreducibility is far more difficult to prove). A product of fundamental representations of definite permutation symmetry is written as

$$\sum_{ijk\dots} C^{ijk\dots} q_i(1) q_j(2) q_k(3) \dots q_m(n)$$

with a definite permutation symmetry incorporated in  $C^{ijk\dots}$ . It transforms into

$$\sum_{\substack{ijk\dots \\ pqr\dots}} C^{ijk\dots} U_i^p U_j^q U_k^r \dots q_p(1) q_q(2) q_r(3) \dots q_s(n) .$$

However all the rotation matrices  $U$  are identical and therefore the coefficient

$$C^{pqr\dots} = C^{ijk\dots} U_i^p U_j^q U_k^r \dots$$

possesses the same permutation symmetry as  $C^{ijk\dots}$ , itself.

The following theorem (not independent of the one above) is also useful in physics.

#### Theorem

- In order to construct irreducible representations from  $q_i q_j q_k \dots \bar{q}^p \bar{q}^q \bar{q}^r \dots$ ,
- (1) symmetrize and/or antisymmetrize according to the Young tableau rule in  $(ijk\dots)$  and  $(pqr\dots)$  separately, and
  - (2) Separate invariant subspaces and subspaces of lower dimensions by taking traces in pairs of upper and lower indices.

Example 1  $\bar{q}^i \otimes q_j = \frac{1}{3} \delta_j^i \text{Tr}(\bar{q} q) + \left( \bar{q}^i q_j - \frac{1}{3} \delta_j^i \text{Tr}(\bar{q} q) \right) \rightarrow \bar{3} \otimes 3 = \underline{1} + \underline{8}.$

Example 2  $\left( \bar{q}^i q_k - \frac{1}{3} \delta_k^i \text{Tr}(\bar{q} q) \right) \otimes \left( \bar{q}^j q_m - \frac{1}{3} \delta_m^j \text{Tr}(\bar{q} q) \right) = T_k^i \otimes S_m^j$  with  $T_1^i = S_j^j = 0.$

- (1)  $T_j^i S_1^j = \text{invariant. } \underline{1}$
- (2)  $(T_j^i S_m^j - T_m^j S_j^i) = \begin{array}{|c|} \hline \square \\ \hline \end{array}$  or  $\begin{array}{|c|} \hline \square \\ \hline \end{array}$  and one contraction.  $\underline{8}_A$  (antisym. under  $T \leftrightarrow S$ )
- (3)  $(T_j^i S_m^j + T_m^j S_j^i - \frac{2}{3} \delta_m^i \text{Tr}(TS)) = \begin{array}{|c|} \hline \square \\ \hline \end{array}$  or  $\begin{array}{|c|} \hline \square \\ \hline \end{array}$  with one contraction.  $\underline{8}_S$   
(equivalent to  $\underline{8}_A$ )
- (4)  $\begin{array}{|c|} \hline \square \\ \hline \end{array}$  and no contraction, but traces subtracted;
- $$T_k^i S_m^j - T_m^j S_k^i + T_m^i S_k^j - T_k^j S_m^i - \frac{1}{3} \delta_m^i (T_k^n S_n^j - T_n^j S_k^n) - \frac{1}{3} \delta_k^i (T_m^n S_n^j - T_n^j S_m^n) \\ - \frac{1}{3} \delta_k^j (T_m^n S_n^i - T_n^n S_m^i) - \frac{1}{3} \delta_m^j (T_k^n S_n^i - T_n^n S_k^i).$$

This is 10 dimensional representation.

- (5)  $\begin{array}{|c|} \hline \square \\ \hline \end{array}$  and no contraction, but traces subtracted.

This is 10 representation (inequivalent to 10, just like 3 inequivalent to 3).

- (6)  $\begin{array}{|c|} \hline \square \\ \hline \end{array}$  and no contraction, but traces subtracted. This is 27.

Therefore,

$$\underline{8} \otimes \underline{8} = \underline{1} + \underline{8}_A + \underline{8}_S + \underline{10} + \overline{10} + \underline{27}.$$

Among these, (1,  $\underline{8}_S$ , 27) are even under  $T \leftrightarrow S$ , while ( $\underline{8}_A$ , 10, 10) are odd under  $T \leftrightarrow S$ .

#### Remarks on the Young tableau rules

- (1) # of boxes in the n-th row does not exceed # of boxes in the (n-1)-th row.
- (2) # of boxes in the n-th column does not exceed # of boxes in the (n-1)-th column.
- (3) Label boxes by particles (instead of states which particles occupy) in the ascending order from top to bottom and from left to right. In this way you find how many equivalent representations exist.
- (4) First symmetrize states of particles within each row and then antisymmetrize states of particles within each column.

#### 2.2b Particle classification

Conserved quantum numbers: Two of  $\lambda_a$  ( $a = 1, 2, \dots, 8$ ) are simultaneously diagonalizable (it is called that the SU(3) group has rank 2). We normally diagonalize  $\lambda_3$  and  $\lambda_8$  as shown before.  $\frac{1}{2} \lambda_3$  is identified with the third component of isospin.  $\frac{1}{\sqrt{3}} \lambda_8$  is called the hypercharge Y.  $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$  are eigenstates of  $I_3$  and Y with eigenvalues  $I_3(u) = 1/2$ ,  $I_3(d) = -1/2$ ,  $I_3(s) = 0$ ,  $Y(u) = 1/3$ ,  $Y(d) = 1/3$ , and  $Y(s) = -2/3$ .  $\begin{pmatrix} \bar{u} \\ \bar{d} \\ \bar{s} \end{pmatrix}$  are also eigenstates of  $I_3$  and Y, but with eigenvalues opposite in sign,  $I_3(\bar{u}) = -1/2$ ,  $I_3(\bar{d}) = 1/2$ ,  $I_3(\bar{s}) = 0$ ,  $Y(\bar{u}) = -1/3$ ,  $Y(\bar{d}) = -1/3$ ,  $Y(\bar{s}) = 2/3$ . One can define strangeness, if one wishes, as  $S = \frac{1}{\sqrt{3}} \lambda_8 - \frac{1}{3} \mathbf{1} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$  for quarks  $\begin{pmatrix} u \\ d \\ s \end{pmatrix}$ . The electric charges of q's are to be determined from the requirement that the baryons made of qqq have integral charges. Then we find

$$Q = I_3 + \frac{1}{2} Y = \frac{1}{2} \lambda_3 + \frac{1}{2\sqrt{3}} \lambda_8 = \begin{pmatrix} 2/3 & -1/3 & -1/3 \end{pmatrix}.$$

The operators of the "SU(3) charges" in terms of creation-annihilation operators of particles are given by  $\int d^3x : \bar{q} \frac{1}{2} \lambda_3 \gamma_0 q = \sum_{\vec{p}, s} (b_{\vec{p}, s}^\dagger(q) \frac{1}{2} \lambda_3 b_{\vec{p}, s}(q) - d_{\vec{p}, s}^\dagger(q) \frac{1}{2} \lambda_3 d_{\vec{p}, s}(q))$ .

Therefore, the value of  $I_3$  for the one-u-quark state, for instance, is given as

$$I_3 |u(\vec{p}, s)\rangle = \sum_{\vec{p}', s'} (b_{\vec{p}', s'}^\dagger(q) \frac{1}{2} \lambda_3 b_{\vec{p}', s'} - d_{\vec{p}', s'}^\dagger(q) \frac{1}{2} \lambda_3 d_{\vec{p}', s'}) b_{\vec{p}, s}^\dagger(u) |0\rangle = +\frac{1}{2} b_{\vec{p}, s}^\dagger(u) |0\rangle.$$

We assign hadrons with the same  $J^P$  and with (approximately) the same mass into an irreducible representation.

Mesons (made of  $\bar{q}q$ ):

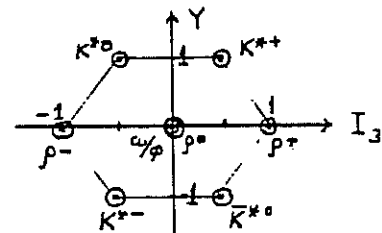
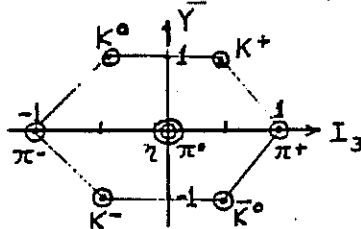
Singlets  $\bar{q}q_k/\sqrt{3}$ . For instance,  $\eta'(0^-, 958 \text{ MeV})$

By the obvious reason,  $I(\underline{1}) = 0$  and  $Y(\underline{1}) = 0$ .

Octets.  $\bar{q}^j q_i - \frac{1}{3} \delta_i^j (\bar{q}q) \sim \bar{q} \lambda_a q$  ( $i, j = 1, 2, 3; a=1, 2, \dots, 8$ )

	$I_3$	$I$	$Y$	$Q$	meson states of $1S_0$ [ $3S_1$ ]
$\bar{s}u = \frac{1}{2} \bar{q}(\lambda_4 - i\lambda_5)q$	$1/2$	$1/2$	$1$	$1$	$K^+(495 \text{ MeV})$ [ $K^{*+}(892 \text{ MeV})$ ]
$\bar{s}d = \frac{1}{2} \bar{q}(\lambda_6 - i\lambda_7)q$	$-1/2$			$0$	$K^0(500 \text{ MeV})$ [ $K^{*0}(892 \text{ MeV})$ ]
$\bar{d}u = \frac{1}{2} \bar{q}(\lambda_1 - i\lambda_2)q$	$1$	$1$	$0$	$1$	$\pi^+(140 \text{ MeV})$ [ $\rho^+(770 \text{ MeV})$ ]
$\frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d) = \frac{1}{\sqrt{2}} \bar{q}\lambda_3 q$	$0$			$0$	$\pi^0(135 \text{ MeV})$ [ $\rho^0(770 \text{ MeV})$ ]
$\bar{u}d = \frac{1}{2} \bar{q}(\lambda_1 + i\lambda_2)q$	$-1$			$-1$	$\pi^-(140 \text{ MeV})$ [ $\rho^-(770 \text{ MeV})$ ]
$\bar{d}s = \frac{1}{2} \bar{q}(\lambda_6 + i\lambda_7)q$	$1/2$	$1/2$	$-1$	$0$	$\bar{K}^0(500 \text{ MeV})$ [ $\bar{K}^{*0}(892 \text{ MeV})$ ]
$\bar{u}s = \frac{1}{2} \bar{q}(\lambda_4 + i\lambda_5)q$	$-1/2$			$-1$	$K^-(495 \text{ MeV})$ [ $K^{*-}(892 \text{ MeV})$ ]
$\frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s) = \frac{1}{\sqrt{2}} \bar{q}\lambda_8 q$	$0$	$0$	$0$	$0$	$\eta(549 \text{ MeV})$ [ $\omega(783 \text{ MeV})$ ] [ $\phi(1020 \text{ MeV})$ ]?

The eight states of 8 can be plotted with  $I_3$  in x-axis and  $Y$  in y-axis as



The eight states are also written in  $(3 \times 3)$  matrix with the columns referring to the indices of  $\bar{q}$  and the rows referring to the indices of  $q$ .

$$M_J^1 = \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}}, & \pi^+, & K^+ \\ \pi^-, & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta}{\sqrt{6}}, & K^0 \\ K^-, & \bar{K}^0, & -\frac{2}{\sqrt{6}}\eta \end{pmatrix} = \frac{1}{\sqrt{2}} \lambda_a M_a, \quad \begin{pmatrix} \frac{\rho^0}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}}, & \rho^+, & K^{*+} \\ \rho^-, & -\frac{\rho^0}{\sqrt{2}} + \frac{\phi_8}{\sqrt{6}}, & K^{*0} \\ K^{*-}, & \bar{K}^{*0}, & -\frac{2}{\sqrt{6}}\phi_8 \end{pmatrix} \text{ for } 1^-.$$

Baryons (made of  $qqq$ ):

Singlets.  $\epsilon^{ijk} q_i q_j q_k$ .  $I = Y = 0$ .  $\Lambda^1$  or  $Y^{0*} (\frac{1}{2}, 1405 \text{ MeV})$ .

Strangeness need not be 0 for  $\underline{1}$ .

Octets.  $q_i q_j q_k + q_j q_i q_k - q_k q_j q_i - q_k q_i q_j$ ,

$q_i q_j q_k + q_k q_j q_i - q_j q_i q_k - q_j q_k q_i$ , or their linear combinations.

	$I_3$	$I$	$Y$	$Q$	baryon states of $J^P = \frac{1}{2}^+$
$\frac{1}{\sqrt{2}}(udu - duu) \sim \bar{s}u$	$1/2$	$\left. \begin{array}{l} 1/2 \\ -1/2 \end{array} \right\}$	$1/2$	$1$	$p$ (938 MeV)
$\frac{1}{\sqrt{2}}(udd - dud) \sim \bar{s}d$	$-1/2$				$n$ (939 MeV)
$\frac{1}{\sqrt{2}}(suu - usu) \sim \bar{d}u$	$1$	$\left. \begin{array}{l} 1 \\ 0 \\ -1 \end{array} \right\}$	$1$	$0$	$\Sigma^+$ (1189 MeV)
$\frac{1}{2}(dsu - sdu - sud + usd) \sim \frac{1}{\sqrt{2}}(\bar{u}u - \bar{d}d)$	$0$				$\Sigma^0$ (1192 MeV)
$\frac{1}{\sqrt{2}}(dsd - sdd) \sim \bar{d}d$	$-1$				$\Sigma^-$ (1197 MeV)
$\frac{1}{\sqrt{2}}(sus - uss) \sim \bar{d}s$	$1/2$	$\left. \begin{array}{l} 1/2 \\ -1/2 \end{array} \right\}$	$1/2$	$-1$	$\Xi^0$ (1315 MeV)
$\frac{1}{\sqrt{2}}(dss - sds) \sim \bar{u}s$	$-1/2$				$\Xi^-$ (1321 MeV)
$\frac{1}{\sqrt{2}}(dsu - sdu + sud - usd - 2uds + 2dus) \sim \frac{1}{\sqrt{6}}(\bar{u}u + \bar{d}d - 2\bar{s}s)$	$0$	$0$	$0$	$0$	$\Lambda$ (1116 MeV)

Although the baryon octet is made of  $qqq$ , they transform exactly like the meson octet with the appropriate correspondence written in the second column above. We can write therefore the baryon octet in the form of

$$B_j^i = \begin{pmatrix} \frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & \Sigma^+ & p \\ \Sigma^- & -\frac{\Sigma^0}{\sqrt{2}} + \frac{\Lambda}{\sqrt{6}} & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{pmatrix},$$

where the rows refer to the third  $q$  of  $qqq$ , while the columns refer to  $\epsilon^{ikm}q_k(1)q_m(2)$ .

Decuplet (totally symmetric  $qqq$ ):

	$I_3$	$I$	$Y$	$Q$	baryon states of $J^P = \frac{3}{2}^+$
$uuu$	$3/2$	$\left. \begin{array}{l} 3/2 \\ 1/2 \\ -1/2 \\ -3/2 \end{array} \right\}$	$1$	$2$	$\Delta^{++}$ (1232 MeV)
$\frac{1}{\sqrt{3}}(uud + udu + duu)$	$1/2$			$1$	$\Delta^+$ (1232 MeV)
$\frac{1}{\sqrt{3}}(udd + dud + ddu)$	$-1/2$			$0$	$\Delta^0$ (1232 MeV)
$ddd$	$-3/2$			$-1$	$\Delta^-$ (1232 MeV)
$\frac{1}{\sqrt{3}}(uus + usu + suu)$	$1$	$\left. \begin{array}{l} 1 \\ 0 \\ -1 \end{array} \right\}$	$1$	$1$	$\Sigma'^+$ (1382 MeV)
$\frac{1}{\sqrt{6}}(dus + uds + dsu + sdu + usd + sud)$	$0$			$0$	$\Sigma'^0$ (1382 MeV)
$\frac{1}{\sqrt{3}}(dds + dsd + sdd)$	$-1$			$-1$	$\Sigma'^-$ (1382 MeV)
$\frac{1}{\sqrt{3}}(uss + sus + ssu)$	$1/2$	$\left. \begin{array}{l} 1/2 \\ -1/2 \end{array} \right\}$	$1/2$	$0$	$\Xi'^0$ (1533 MeV)
$\frac{1}{\sqrt{3}}(dss + sds + ssd)$	$-1/2$			$-1$	$\Xi'^-$ (1533 MeV)
$sss$	$0$	$0$	$-2$	$-1$	$\Omega^-$ (1672 MeV)

### Antiparticles

For the meson octets, antiparticles are contained in the same representation as their particles. In this sense, the meson octets are called "self-charge conjugate";

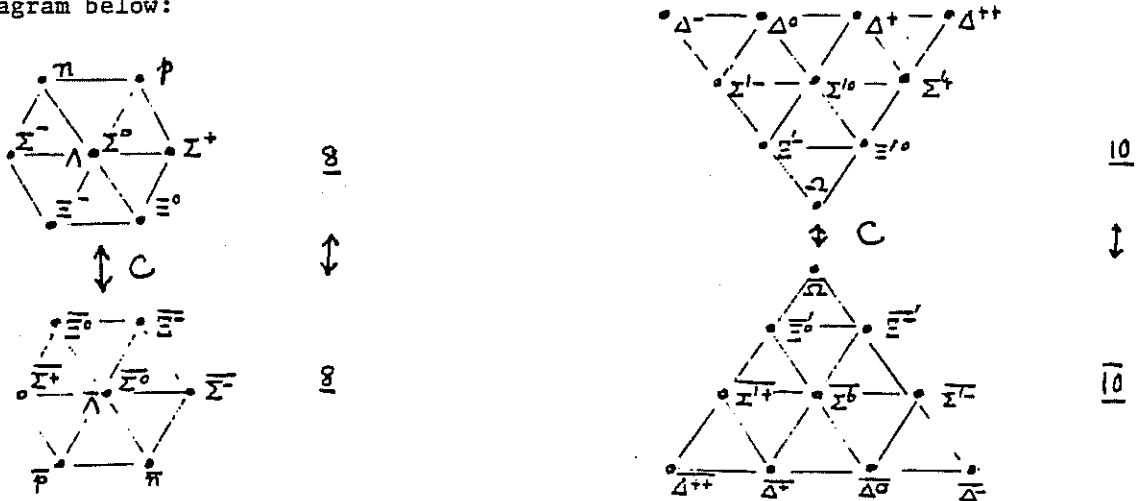
$$M_j^i \xrightarrow{C} \gamma^{(C)} M_i^j.$$

On the other hand, antibaryons make another representation (conjugate to the representation of particles).

$$B_j^i \xrightarrow{C} \gamma^{(C)} \bar{B}_i^j; \quad \bar{B}_j^i = \begin{pmatrix} \bar{\Sigma}^0/\sqrt{2} + \bar{\Lambda}/\sqrt{6}, & \bar{\Sigma}^-, & \bar{\Xi}^- \\ \bar{\Sigma}^+, & -\bar{\Sigma}^0/\sqrt{2} + \bar{\Lambda}/\sqrt{6}, & \bar{\Xi}^0 \\ \bar{p}, & \bar{n}, & -2\bar{\Lambda}/\sqrt{6} \end{pmatrix}.$$

(qqq)                      (qqq)

The antibaryons of  $\underline{10}$  resonances ( $\Delta, \Sigma', \Xi', \Omega$ ) make  $\overline{10}$  representation, as shown in the  $I_3$ -Y diagram below:



### 2.2c SU(3) invariant interactions

Recall how we constructed the SU(2) invariant pion-nucleon Yukawa coupling :

$$\mathcal{N}_{\text{int}} = i g \bar{N} \gamma_5 \vec{\tau} N \cdot \vec{\phi},$$

where  $\bar{N} = (\bar{p}, \bar{n})$ ,  $N = \begin{pmatrix} p \\ n \end{pmatrix}$  so that  $\bar{N} \gamma_5 \vec{\tau} N$  is an isovector ( $\underline{3}$  representation of SU(2)).  $\vec{\phi}$  is another isovector ( $(\pi^+ + \pi^-)/\sqrt{2}, \pi^0, i(\pi^+ - \pi^-)/\sqrt{2}$ ) which transforms just like  $\vec{\tau}$ .

The same pion-nucleon interaction can be written in the form of

$$\mathcal{N}_{\text{int}} = \sqrt{2} \sum_{ij} i g \bar{N}^i \gamma_5 N_j \phi_i^j \quad \text{with} \quad \phi_i^j = \begin{pmatrix} \pi^0/\sqrt{2} & \pi^+ \\ \pi^- & -\pi^0/\sqrt{2} \end{pmatrix}$$

Through the correspondence  $\tau_a (a=1,2,3) \leftrightarrow \lambda_a (a=1,2,\dots,8)$ , it is straightforward to construct the SU(3) invariant meson-quark Yukawa coupling as

$$\mathcal{N}_{\text{int}} = f \sum_a \bar{q} \Gamma \lambda_a q M_a = \sqrt{2} f \sum_{ij} \bar{q}^i \Gamma q_j M_j^i \quad \text{with} \quad M_j^i = \sum_a \left( \frac{\lambda_a}{\sqrt{2}} \right)_j^i M_a.$$

where  $\Gamma$  is  $i\gamma_5$  ( $\gamma_\mu$ ) for  $M = J^P = 0^-$  ( $1^-$ ).

For the meson-baryon ( $\underline{8}$ ) Yukawa couplings, there are more SU(3) indices and therefore more than one SU(3) invariant couplings in general. Let us do a few exercises.

(a)  $\bar{B}(\underline{8})B(\underline{8})M(\underline{1})$ :  $g \bar{B}_j^i B_i^j M = g \text{Tr}(\bar{B}B) M.$

Note that  $\text{Tr} B = \text{Tr} \bar{B} = 0$  and  $M(\underline{1})_j^i = \delta_j^i M.$

(b)  $\bar{B}(\underline{8})B(\underline{8})M(\underline{8})$ : Recall the decomposition of  $\underline{8} \otimes \underline{8} = \underline{1} + \underline{8}_A + \underline{8}_S + \underline{10} + \overline{10} + \underline{27}$

There are two different ways to make  $\underline{8}$  from  $\bar{B}(\underline{8})$  and  $B(\underline{8})$  to match  $M(\underline{8})$ . Therefore, there are two independent Yukawa couplings as

$$\begin{aligned}\mathcal{K}_{\text{int}} &= f \text{Tr}(\bar{B} M B) + g \text{Tr}(\bar{B} B M) = f \bar{B}_j^i M_i^k M_k^j + g \bar{B}_j^i B_i^k M_k^j \\ &= g_D \left( \text{Tr}(\bar{B} M B) + \text{Tr}(\bar{B} B M) \right) + g_F \left( \text{Tr}(\bar{B} M B) - \text{Tr}(\bar{B} B M) \right).\end{aligned}$$

The coupling  $g_D$  is symmetric under interchange of  $B$  and  $\bar{B}$  (not counting the anticommutativity of fields) and the  $g_F$  coupling is antisymmetric. Another way of writing these Yukawa couplings is to use the expression

$$\bar{B}_j^i = \sum_a \frac{\lambda_a}{\sqrt{2}} \bar{B}_a, \quad B_j^i = \sum_a \frac{\lambda_a}{\sqrt{2}} B_a, \quad M_j^i = \sum_a \frac{\lambda_a}{\sqrt{2}} M_a \quad (a \text{ summed over } 1, 2, \dots, 8)$$

Then,

$$\mathcal{K}_{\text{int}} = \sqrt{2} g_D d_{abc} \bar{B}_a B_b M_c - i\sqrt{2} g_F f_{abc} \bar{B}_a B_b M_c \quad (a, b, c \text{ summed over } 1, 2, \dots, 8)$$

From this form of coupling, it is easy to read off

$$\mathcal{K}_{\text{int}} = i (g_D + g_F) \bar{\psi} \gamma_5 \pi^+ - i\sqrt{2} \left( \frac{1}{\sqrt{6}} g_D + \sqrt{\frac{3}{2}} g_F \right) \bar{\Lambda} \gamma_5 p K^- + \dots$$

From experimental determination, we know that  $g_D/g_F = 1.5 \sim 2.0$ .

#### (c) $M(\underline{8}) M'(\underline{8}) M''(\underline{8})$ couplings

The  $SU(3)$  group structure is identical to that of  $\bar{B}(\underline{8})B(\underline{8})M(\underline{8})$ . Therefore, we expect two independent couplings most generally from group theory. However, if these meson octets are self-charge conjugate, an extra condition is imposed on the couplings (and one of the two coupling must go away).

Under  $C$  conjugation  $C M_j^i C^{-1} = \eta^{(C)} M_i^j$  ( $\eta^{(C)}$  is common to all 8 components), the coupling

$$\begin{aligned}\mathcal{K}_{\text{int}} &= f_D \left( \text{Tr}(M(1)M(2)M(3)) + \text{Tr}(M(1)M(3)M(2)) \right) \\ &\quad + f_F \left( \text{Tr}(M(1)M(2)M(3)) - \text{Tr}(M(1)M(3)M(2)) \right) = \mathcal{K}^{(D)} + \mathcal{K}^{(F)}\end{aligned}$$

transforms as

$$C \mathcal{K}_{\text{int}} C^{-1} = \eta_1^{(C)} \eta_2^{(C)} \eta_3^{(C)} \left( \mathcal{K}^{(D)} - \mathcal{K}^{(F)} \right).$$

If  $\eta_1^{(C)} \eta_2^{(C)} \eta_3^{(C)} = +1$ , then  $f_D \neq 0$  and  $f_F = 0$ . This is the case for the  $2^{++}0^-0^-$  coupling with  $2^{++} = (A_2(1318), K^*(1434), f/f'(1273/1520))$ . If  $\eta_1^{(C)} \eta_2^{(C)} \eta_3^{(C)} = -1$ , then  $f_D = 0$  and  $f_F \neq 0$ . This is the case for the  $1^{--}0^-0^-$  coupling. It should be remarked here that from the Lorentz invariance and the subsidiary condition on the spin 2 fields ( $\partial^\mu T_{\mu\nu} = \partial^\nu T_{\mu\nu} = 0$ ) the  $2-0-0$  coupling has to be of the form of

$$g_{abc} \partial_\mu \phi_a \partial_\nu \phi_b T_c^{\mu\nu} \quad (\text{symmetric under interchange of } a \text{ and } b),$$

$T^{\mu\nu} = T^{\nu\mu}$

while the  $1-0-0$  coupling should be of the form of

$$g'_{abc} (\phi_a \partial_\mu \phi_b - \partial_\mu \phi_a \phi_b) V_c^\mu \quad (\text{antisymmetric under } a \leftrightarrow b).$$

Check by yourself that there is no constraint imposed on the meson-baryon coupling  $\bar{B}(\underline{8})B(\underline{8})M(\underline{8})$

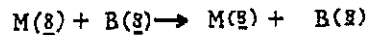
(d)  $\bar{B}(\underline{10})B(\underline{8})M(\underline{8})$ : ( $\Delta \leftrightarrow \pi N$ ,  $\Sigma' \leftrightarrow \pi \Lambda$  etc decay couplings)

$\bar{B}(\underline{3})M(\underline{6})$  can make only one  $\underline{10}$  (the other one is  $\underline{\bar{10}}$ ), so there is only one independent  $SU(3)$  coupling.

$$\mathcal{K}_{\text{int}} = g \varepsilon_{ijk} \bar{B}^{imn} B_m^j M_n^k + \text{h.c.}$$

(e) Two-body scattering amplitudes (or effective four-body interactions)

$S_{fi} = \langle f | S | i \rangle$  with  $S = \text{SU}(3)$  singlet (ignoring  $\text{SU}(3)$  breaking interactions). In order that  $S_{fi} \neq 0$ ,  $|i\rangle$  and  $|f\rangle$  must belong to the same  $\text{SU}(3)$  representation, or equivalently,  $\langle f|$  and  $|i\rangle$  must be able to form an  $\text{SU}(3)$  singlet. The  $\text{SU}(3)$  structure of the scattering



is given by

$$\begin{aligned} \mathcal{M} = & a_1 \text{Tr}(\bar{B} B M_i \bar{M}_f) + a_2 \text{Tr}(\bar{B} M_f M_i B) + a_3 \text{Tr}(\bar{B} B M_f M_i) + a_4 \text{Tr}(\bar{B} M_f B M_i) + a_5 \text{Tr}(\bar{B} M_i B M_f) \\ & + a_6 \text{Tr}(\bar{B} M_i M_f B) + a_7 \text{Tr}(\bar{B} B) \text{Tr}(M_i \bar{M}_f) + a_8 \text{Tr}(\bar{B} M_f) \text{Tr}(B M_i) + a_9 \text{Tr}(\bar{B} M_i) \text{Tr}(B M_f). \end{aligned}$$

Here, I put the bars on the final particles because they refer to the creation operators which transform just like the annihilation operators of their antiparticles. In fact, not all of the 9 amplitudes are independent, as you see from

$$\begin{aligned} |B \times \bar{M}_f\rangle &= \frac{1}{1} + \frac{8_A}{8} + \frac{8_S}{8} + \frac{10}{10} + \frac{10}{10} + \frac{27}{27}, \\ |B \times M_i\rangle &= \frac{1}{1} + \frac{8_A}{8} + \frac{8_S}{8} + \frac{10}{10} + \frac{10}{10} + \frac{27}{27}. \end{aligned}$$

One of  $a_1 \sim a_9$  is dependent of the others.

## 2.2d SU(3) Clebsch-Gordan coefficients

It is, in principle, straightforward to figure out the  $\text{SU}(3)$  relations among couplings and amplitudes from the tensor analysis given above. However, there is a short cut in this analysis if you know something equivalent to the Clebsch-Gordan coefficients of the rotation group. The  $\text{SU}(3)$  Clebsch-Gordan coefficients are tabulated for the products of  $\text{SU}(3)$  representations which appear frequently in particle physics. I will present them in the form of de Swart (the same as those tabulated in the Particle Data Table).

First specify the states by  $\text{SU}(3)$  representation  $n$  ( $=1, 8, 10, \bar{10}, 27, \dots$ ),  $I$ ,  $I_3$ , and  $Y$ . The the  $\text{SU}(3)$  C.G. coefficients are defined exactly like the  $\text{SU}(2)$  C.G. coefficients;

$$\begin{aligned} & |n, I, I_3, Y\rangle |n', I', I'_3, Y'\rangle \\ &= \sum_{n'', I''} \langle n'', I'', I_3+I'_3, Y+Y' | n, I, I_3, Y; n', I', I'_3, Y' \rangle |n'', I'', I_3+I'_3, Y+Y'\rangle \end{aligned}$$

The C.G. coefficients above are given, for instance, in MacNamee and Chilton, Rev. of Mod. Phys. 36, 1005(1964). The tables are quite large in size. It is possible to separate these C.G. coefficients into two parts, one that depends only on  $(n, n', n'')$  and  $(I, I', I'')$  and the other that is the  $\text{SU}(2)$  C.G. coefficients (dependent only on  $(I, I', I'')$  and  $(I_3, I'_3)$ ). The former part of the  $\text{SU}(3)$  C.G. coefficients is called the isoscalar factor in the sense that it does not depend on the third components of isospins. If you express the C.G. coefficients in this way, the tables are much shorter because you already have the  $\text{SU}(2)$  C.G. coefficients. They were tabulated by J. J. de Swart in Rev. Mod. Phys. 35, 916 (1964) and also in the Particle Data table. Take, for instance, the product of  $\pi^-$  and  $p$  (proton):

$$|\pi^-\rangle \times |p\rangle = |8, 1, -1, 0\rangle \times |8, \frac{1}{2}, \frac{1}{2}, 1\rangle.$$

$$|p\pi^-\rangle = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{1}}{3}\right)|\underline{27}, \frac{3}{2}, -\frac{1}{2}\rangle + \left(\frac{\sqrt{5}}{10}\right)\left(\frac{\sqrt{2}}{3}\right)|\underline{27}, \frac{1}{2}, -\frac{1}{2}\rangle + \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{1}}{3}\right)|\underline{10}, \frac{3}{2}, -\frac{1}{2}\rangle \\ + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{3}\right)|\underline{10}, \frac{1}{2}, -\frac{1}{2}\rangle + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{3}\right)|\underline{8}_A, \frac{1}{2}, -\frac{1}{2}\rangle + \left(\frac{3\sqrt{5}}{10}\right)\left(\frac{\sqrt{2}}{3}\right)|\underline{8}_S, \frac{1}{2}, -\frac{1}{2}\rangle.$$

all Y = +1.

(Note the SU(2) C.G. sign convention (which appears in p.2.18) for the second factors.)

## SU(2) CLEBSCH-GORDAN COEFFICIENTS AND SPHERICAL HARMONICS

Note: A  $\sqrt{\quad}$  is to be understood over every coefficient; e.g., for  $-8/15$  read  $-\sqrt{8/15}$ .

Notation:

$J_1$	$J_2$	$J_3$	$M_1$	$M_2$	$M_3$
$m_1$	$m_2$		$m_1$	$m_2$	
Coefficients					

$1/2 \times 1/2$

$1/2$	$1/2$	$1$	$0$	$0$
$+1/2$	$+1/2$	$1$	$0$	$0$
$-1/2$	$-1/2$	$1$	$0$	$0$
$+1/2$	$-1/2$	$1$	$0$	$0$
$-1/2$	$+1/2$	$1$	$0$	$0$

$Y_1^0 = \sqrt{\frac{3}{4\pi}} \cos\theta$

$Y_1^1 = -\sqrt{\frac{3}{8\pi}} \sin\theta e^{i\phi}$

$2 \times 1/2$

$5/2$	$3/2$	$1$	$0$	$0$
$+5/2$	$+3/2$	$1$	$0$	$0$
$-5/2$	$-3/2$	$1$	$0$	$0$
$+3/2$	$+1/2$	$1$	$0$	$0$
$-3/2$	$-1/2$	$1$	$0$	$0$

$Y_2^0 = \sqrt{\frac{5}{4\pi}} \left( \frac{3}{2} \cos^2\theta - \frac{1}{2} \right)$

$Y_2^1 = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta e^{i\phi}$

$Y_2^2 = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2\theta e^{2i\phi}$

$1 \times 1/2$

$3/2$	$1/2$	$1$	$0$	$0$
$+3/2$	$+1/2$	$1$	$0$	$0$
$-3/2$	$-1/2$	$1$	$0$	$0$
$+1/2$	$-1/2$	$1$	$0$	$0$
$-1/2$	$+1/2$	$1$	$0$	$0$

$3/2 \times 1/2$

$5/2$	$3/2$	$1$	$0$	$0$
$+5/2$	$+3/2$	$1$	$0$	$0$
$-5/2$	$-3/2$	$1$	$0$	$0$
$+3/2$	$+1/2$	$1$	$0$	$0$
$-3/2$	$-1/2$	$1$	$0$	$0$

$2 \times 1$

$3$	$2$	$1$	$0$	$0$
$+3$	$+2$	$1$	$0$	$0$
$-3$	$-2$	$1$	$0$	$0$
$+1$	$-1$	$1$	$0$	$0$
$-1$	$+1$	$1$	$0$	$0$

$3/2 \times 1$

$5/2$	$3/2$	$1$	$0$	$0$
$+5/2$	$+3/2$	$1$	$0$	$0$
$-5/2$	$-3/2$	$1$	$0$	$0$
$+3/2$	$+1/2$	$1$	$0$	$0$
$-3/2$	$-1/2$	$1$	$0$	$0$

$1 \times 1$

$2$	$1$	$0$	$0$	$0$
$+2$	$+1$	$0$	$0$	$0$
$-2$	$-1$	$0$	$0$	$0$
$+1$	$-1$	$0$	$0$	$0$
$-1$	$+1$	$0$	$0$	$0$

$Y_l^{-m} = (-1)^m Y_l^m$

$(j_1 j_2 m_1 m_2 | j_3 j_3 M)$

$= (-1)^{j_1 - j_2 - j_3} (j_2 j_1 m_2 m_1 | j_3 j_3 M)$

## SU(3) CONVENTIONS

for Isoscalar Factor Table on next page

Since January 1970 we have used the convention that the first particle shall be a baryon, the second a meson (R. Levi Setti, Proceedings of Lund Conference, 1969, p. 339 and Table II). Note, for comparison, that the de Swart table of  $8 \times 8$  is merely labeled with symbols like  $(I_1 = 1/2, Y_1 = 1, I_2 = 1, Y_2 = 0)$ , which can be read either as  $(N\pi)$  or  $(K\pi)$ . Since there are no decuplet mesons, however, his  $8 \times 10$  table is unambiguous; it must be read with the meson first.

The de Swart convention violates the other convention that the  $N, N\pi$  coupling shall be  $D + F$  (as opposed to  $-D + F$ ). To get  $D + F$  one must use the first line of the "N" table, which reads. . .  $3\sqrt{5}/10 |8_D\rangle + 1/2 |8_F\rangle$  as opposed to. . .  $-3\sqrt{5}/10 |8_D\rangle + 1/2 |8_F\rangle$ . The first line must then be labeled  $N\pi$  rather than  $K\pi$ , i.e., with the baryon first.

Levi Setti further advocates the convention of writing the baryon first for SU(2) as well as SU(3). For example, the sign of the amplitudes as plotted on his and our Argand plots comes from using our SU(2) Clebsch-Gordan coefficients (Condon Shortley notation) and writing the baryon first. To make it easier to abide by this universal convention we have changed de Swart's  $8 \times 10$  (SU(3) table to  $10 \times 8$ , with the help of his Eq. (14.3):

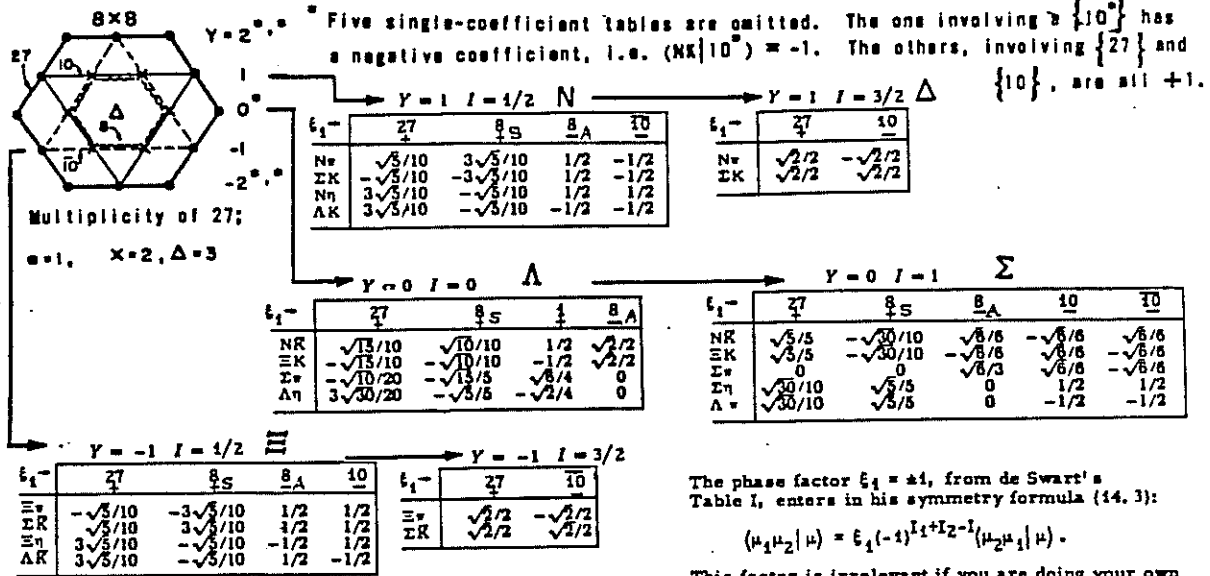
(continued from the previous page)

$$\langle \mu_1 \mu_2 | \mu \rangle = \xi_1 (-1)^{I_1 + I_2 - I} \langle \mu_2 \mu_1 | \mu \rangle \quad \text{for } SU(2) \text{ CG.}$$

## SU(3) ISOSCALAR FACTORS

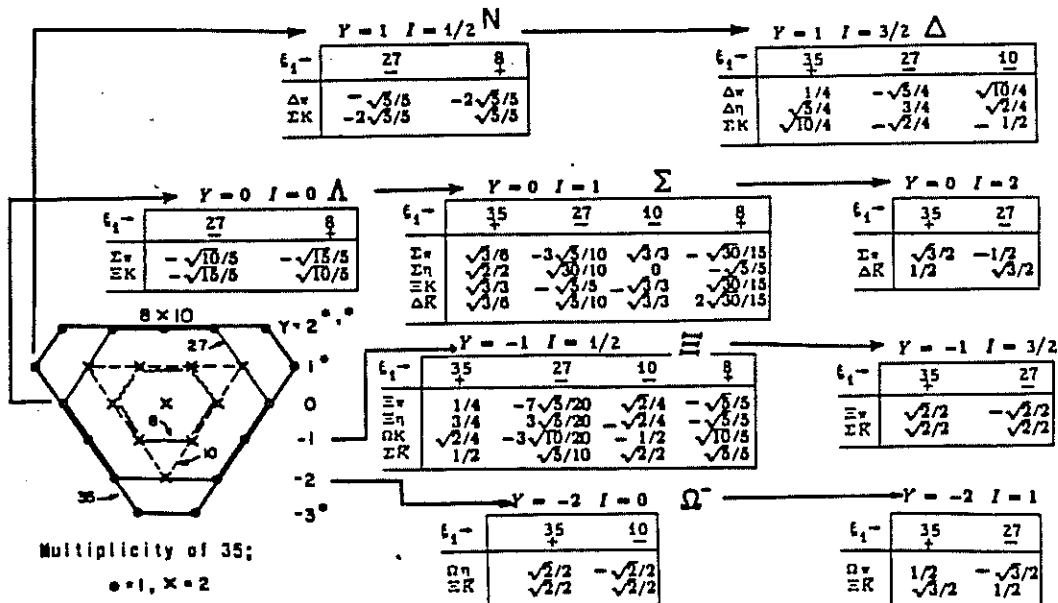
Adapted from J. J. de Swart, Rev. Mod. Phys. **35**, 916 (1963)  
(See note on previous page concerning conventions)

$$[8] \otimes [8] = [27] \oplus [10] \oplus [10^*] \oplus [8]_1 \oplus [8]_2 \oplus [1].$$



$$[10] \otimes [8] = [35] \oplus [27] \oplus [10] \oplus [8].$$

\* Four single coefficient tables are omitted; only the  $\{27\}$  is -1; the three with  $\{35\}$  are +1.



$$H_A = \sqrt{2}g_a [\text{Tr}(\bar{B}MB) - \text{Tr}(\bar{B}BM)] \quad \text{and} \quad H_S = \sqrt{2}g_s [\text{Tr}(\bar{B}MB) + \text{Tr}(\bar{B}BM)]$$

$$g_s = \frac{1}{\sqrt{2}} g_D \quad \text{and} \quad g_a = \frac{1}{\sqrt{2}} g_F \quad \text{in the Note.} \quad \alpha \equiv g_s / (g_s + g_a)$$

TABLE 4.1

Charge independent form	$\text{Tr}(\bar{B}MB)$	$\text{Tr}(\bar{B}BM)$	$H_S$	$H_A$	$H$
$\bar{N}\tau \cdot N\pi$	$\frac{1}{\sqrt{2}}$	0	$g_s$	$g_a$	$g$
$\bar{\Sigma} \cdot \Xi\tau K + \bar{\Xi}\tau \cdot \Sigma K$	$\frac{1}{\sqrt{2}}$	0	$g_s$	$g_a$	$g$
$i\bar{\Sigma} \times \Sigma \cdot \pi$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	0	$-2g_a$	$-2\alpha g$
$\bar{\Lambda}\Sigma \cdot \pi + \bar{\Sigma} \cdot \Lambda\pi$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{2}{\sqrt{3}}g_s$	0	$\frac{2}{3}(1-\alpha)g$
$\bar{\Lambda}\Lambda\eta^0$	$-\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{6}}$	$-\frac{2}{\sqrt{3}}g_s$	0	$-\frac{2}{3}(1-\alpha)g$
$\bar{\Sigma} \cdot \Sigma\eta^0$	$\frac{1}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$\frac{2}{\sqrt{3}}g_s$	0	$\frac{2}{3}(1-\alpha)g$
$\bar{\Xi}\tau\Xi\pi$	0	$\frac{1}{\sqrt{2}}$	$g_s$	$-g_a$	$(1-2\alpha)g$
$\bar{\Sigma}N\tau K + \bar{N}\tau \cdot \Sigma K$	0	$\frac{1}{\sqrt{2}}$	$g_s$	$-g_a$	$(1-2\alpha)g$
$\bar{N}N\eta^0$	$\frac{1}{\sqrt{6}}$	$-\frac{2}{\sqrt{6}}$	$-\frac{1}{\sqrt{3}}g_s$	$\sqrt{3}g_a$	$-\frac{1}{3}(1-4\alpha)g$
$\bar{\Xi}\Lambda K + \bar{\Lambda} \Xi K$	$\frac{1}{\sqrt{6}}$	$-\frac{2}{\sqrt{6}}$	$-\frac{1}{\sqrt{3}}g_s$	$\sqrt{3}g_a$	$-\frac{1}{3}(1-4\alpha)g$
$\bar{\Xi}\Xi\eta$	$-\frac{2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{3}}g_s$	$-\sqrt{3}g_a$	$-\frac{1}{3}(1+2\alpha)g$
$\bar{\Lambda}N\bar{K} + \bar{N}\Lambda K$	$-\frac{2}{\sqrt{6}}$	$\frac{1}{\sqrt{6}}$	$-\frac{1}{\sqrt{3}}g_s$	$-\sqrt{3}g_a$	$-\frac{1}{3}(1+2\alpha)g$

To be exhaustive, we give in table 4.2 the explicit expression of the charge independent forms corresponding to our particular choice of phases as included in the  $T_i^j$  matrices.

TABLE 4.2

$\bar{N}\tau N\pi$	$(\bar{p}p - \bar{n}n)\pi^0 + \sqrt{2}(\bar{n}p\pi^- + \bar{p}n\pi^+)$
$\bar{\Sigma}\Xi\tau K$	$\bar{\Sigma}^0(\Xi^-K^+ - \Xi^0K^0) + \sqrt{2}(\bar{\Sigma}^-\Xi^-K^0 + \bar{\Sigma}^+\Xi^0K^+)$
$\bar{\Xi}\tau \cdot \Sigma\bar{K}$	$(\bar{\Xi}^-K^- - \bar{\Xi}^0\bar{K}^0)\Sigma^0 + \sqrt{2}(\bar{\Xi}^-\bar{K}^0\Sigma^- + \bar{\Xi}^0\bar{K}^-\Sigma^+)$
$i\bar{\Sigma} \times \Sigma \cdot \pi$	$(\bar{\Sigma}^-\Sigma^+ - \bar{\Sigma}^+\Sigma^-)\pi^0 + (\bar{\Sigma}^0\Sigma^+ - \bar{\Sigma}^-\Sigma^0)\pi^- + (\bar{\Sigma}^+\Sigma^0 - \bar{\Sigma}^0\Sigma^+)\pi^+$
$\bar{\Lambda}\Sigma \cdot \pi + \bar{\Sigma} \cdot \Lambda\pi$	$\bar{\Lambda}(\Sigma^+\pi^- + \Sigma^-\pi^+ + \Sigma^0\pi^0) + (\bar{\Sigma}^+\pi^+ + \bar{\Sigma}^-\pi^- + \bar{\Sigma}^0\pi^0)\Lambda$
$\bar{\Lambda}\Lambda\eta^0$	$\bar{\Lambda}\Lambda\eta^0$
$\bar{\Sigma} \cdot \Sigma\eta^0$	$(\bar{\Sigma}^+\Sigma^+ + \bar{\Sigma}^-\Sigma^- + \bar{\Sigma}^0\Sigma^0)\eta^0$
$\bar{\Xi}\tau\Xi\pi$	$(\bar{\Xi}^-\Xi^- - \bar{\Xi}^0\Xi^0)\pi^0 + \sqrt{2}(\bar{\Xi}^-\Xi^-\pi^+ + \bar{\Xi}^0\Xi^-\pi^+)$
$\bar{\Sigma}N\tau\bar{K}$	$\bar{\Sigma}^0(pK^- - n\bar{K}^0) + \sqrt{2}(\bar{\Sigma}^-\bar{n}K^- + \bar{\Sigma}^+p\bar{K}^0)$
$\bar{N}\tau \cdot \Sigma K$	$(\bar{p}K^+ - \bar{n}K^0)\Sigma^0 + \sqrt{2}(\bar{n}K^-\Sigma^+ + \bar{p}\Sigma^+K^0)$
$\bar{N}N\eta^0$	$(\bar{p}p + \bar{n}n)\eta^0$
$\bar{\Lambda}\Xi K + \bar{\Xi}\Lambda\bar{K}$	$\bar{\Lambda}(\Xi^-K^+ + \Xi^0K^0) + (\bar{\Xi}^-K^- + \bar{\Xi}^0\bar{K}^0)\Lambda$
$\bar{\Xi}\Xi\eta^0$	$(\bar{\Xi}^-\Xi^- + \bar{\Xi}^0\Xi^0)\eta^0$
$\bar{\Lambda}N\bar{K} + \bar{N}\Lambda K$	$\bar{\Lambda}(pK^- + n\bar{K}^0) + (\bar{p}K^+ + nK^0)\Lambda$

### 2.2e SU(3) breaking and mass formulas

The SU(3) symmetry is far more approximate than the SU(2) isospin symmetry, as we can understand from the large  $N-\Lambda-\Sigma-\Xi$  mass splitting in contrast to the p-n mass splitting. The origin of the SU(3) symmetry breaking is, to large extent, understood in quantum chromodynamics. At more phenomenological levels, we have to treat SU(3) symmetry breaking interactions as medium-strong interaction (stronger than the electromagnetic and weak interactions) very roughly of the order of 1/10 of the strongest interactions. We know that the SU(3) breaking interactions (other than the weak and electromagnetic interactions) conserve I and Y, and therefore transform like a quantity of  $I = Y = 0$ . The simplest possibility is

$$\mathcal{H}_{\text{int}}(\text{SU(3) breaking}) \sim \lambda_8 \quad \text{or} \quad \frac{1}{\sqrt{3}} (T_1^1 + T_2^2 - 2T_3^3).$$

An explicit example for such "interaction" term is the quark mass terms with  $m_u = m_d \neq m_s$ :

$$\begin{aligned} \mathcal{H}_{\text{int}}(\text{SU(3) breaking}) &= m(\bar{u}u + \bar{d}d) + m_s \bar{s}s, \\ &= \left(\frac{2}{3}m + \frac{1}{3}m_s\right)(\bar{u}u + \bar{d}d + \bar{s}s) \\ &\quad + \left(\frac{1}{3}m - \frac{1}{3}m_s\right)(\bar{u}u + \bar{d}d - 2\bar{s}s) \sim I + c\lambda_8. \end{aligned}$$

It is conceivable that the breaking interaction is not only  $I=0$  of  $\underline{8}$ , but also  $I=0$  of  $\underline{27}$ . Even if  $I=0$  of  $\underline{27}$  does not exist in the Hamiltonian or the Lagrangian, it can be generated in the second order perturbation of  $H_{\text{int}}$  (which is presumably small enough to be ignored in the first approximation). We therefore make the following assumption on the SU(3) breaking:

Assumption:  $\mathcal{H}_{\text{strong}} = \mathcal{H}(\text{SU(3) singlet}) + \mathcal{H}_{\text{int}}(I=Y=0 \text{ of } \underline{8}) + \mathcal{H}_{\text{int}}(I=Y=0 \text{ of } \underline{27})$   
 with  $\mathcal{H}_{\text{int}}(I=Y=0 \text{ of } \underline{27}) \ll \mathcal{H}_{\text{int}}(I=Y=0 \text{ of } \underline{8}) \ll \mathcal{H}(\underline{1})$ .

Mass formulas: Keep only  $\mathcal{H}(I=Y=0 \text{ of } \underline{8})$  for SU(3) breakings. The mass is the expectation value of the total Hamiltonian for a particle at rest:

$$m_\alpha = \langle \alpha^{\text{in}} | \mathcal{H}_{\text{tot}}(0) | \alpha^{\text{in}} \rangle = \langle \alpha | U^\dagger(0, -\infty) [\mathcal{H}_0^{(r)}(0) + \mathcal{H}_8^{(r)}(0)] U(0, -\infty) | \alpha \rangle.$$

The right hand side should have the SU(3) structure as

$$1 + c\lambda_8 \quad \text{or} \quad 1 + \frac{1}{\sqrt{3}}c(T_1^1 + T_2^2 - 2T_3^3).$$

For  $B(\underline{8})$ ,

$$\begin{aligned} \langle B(\underline{8}) | \mathcal{H}_{\text{tot}}(0) | B(\underline{8}) \rangle &\Rightarrow m_0 \text{Tr}(\bar{B}B) + m_1 \text{Tr}(\bar{B}\lambda_8 B) + m_2 \text{Tr}(\bar{B}B\lambda_8), \\ &= (m_0 + \frac{m_1 + m_2}{\sqrt{3}}) \text{Tr}(\bar{B}B) - \sqrt{3}m_1 \bar{B}_i^3 B_3^i - \sqrt{3}m_2 \bar{B}_3^i B_i^3. \end{aligned}$$

Decomposing this expression into each baryon, one finds

$$\left. \begin{aligned} m_N &= m_0' + m_2' \\ m_\Lambda &= m_0' + 2(m_1' + m_2')/3 \\ m_\Sigma &= m_0' \\ m_{\Xi} &= m_0' + m_1' \end{aligned} \right\} \text{with} \begin{cases} m_0' = m_0 + (m_1 + m_2)/\sqrt{3}, \\ m_1' = -\sqrt{3}m_1 \\ m_2' = -\sqrt{3}m_2 \end{cases} \quad \begin{aligned} &[1129 \text{ MeV vs } 1135 \text{ MeV}] \\ &\frac{m_N + m_{\Xi}}{2} = \frac{3m_\Lambda + m_\Sigma}{4} \end{aligned}$$

As is clear from the  $8 \times 8$  decomposition, the  $B_8$  mass formula contains two unknown parameters in addition to the symmetric term.

For  $B_{10}$ , there is only one unknown in addition to the symmetric term because  $10 (B_{10})$  and  $\overline{10} (B_{10})$  can make only one 8.

$$\begin{aligned} \langle \overline{B}_{10}^{in} | \mathcal{X}_{tot}(0) | B_{10}^{in} \rangle &\Rightarrow m_0 \overline{B}^{ijk} B_{ijk} + \frac{m_1}{\sqrt{3}} (\overline{B}^{1jk} B_{1jk} + \overline{B}^{2jk} B_{2jk} - 2 \overline{B}^{3jk} B_{3jk}), \\ &= (m_0 + m_1/\sqrt{3}) \overline{B}^{ijk} B_{ijk} - \sqrt{3} m_1 \overline{B}^{3jk} B_{3jk}. \end{aligned}$$

Here  $\overline{B}^{ijk}$  and  $B_{ijk}$  are totally symmetric under interchange of a pair of indices. For instance,  $B_{111} = \Delta^{++}$ ,  $B_{113} = B_{131} = B_{311} = \Sigma^+/\sqrt{3}$ ; .....

$$\left. \begin{aligned} m_{\Delta} &= m_0' \\ m_{\Sigma'} &= m_0' + \frac{1}{3} m_1' \\ m_{\Sigma''} &= m_0' + \frac{2}{3} m_1' \\ m_{\Omega} &= m_0' + m_1' \end{aligned} \right\} \text{ with } \begin{aligned} m_0' &= m_0 + m_1/\sqrt{3} \\ m_1' &= -\sqrt{3} m_1 \end{aligned}$$

$$\begin{aligned} m_{\Omega} - m_{\Sigma'} &= m_{\Sigma'} - m_{\Sigma''} = m_{\Sigma''} - m_{\Delta} \\ 147 \text{ MeV} & \quad 148 \text{ MeV} \quad 142 \text{ MeV} \end{aligned}$$

The agreement with experiment is again quite good. The same result can be obtained through the SU(3) Clebsch-Gordan coefficients, too.

For  $M(8)$ ,  $\langle M_{8}^{in} | \mathcal{X}_{tot}(0) | M_{8}^{in} \rangle$  has the same SU(3) structure as  $B_8$ . But, the fact that  $\mathcal{X}_8(0)$  is even under charge conjugation imposes a constraint, eliminating one of the two unknown parameters; outgoing!

$$\begin{aligned} \langle \overline{M}_j^i | \mathcal{X}_{tot}(0) | M_i^j \rangle &= \langle C(\overline{M}_j^i) | \mathcal{X}_{tot}(0) | C(M_i^j) \rangle, \\ &= \langle \overline{M}_i^j | \mathcal{X}_{tot}(0) | M_j^i \rangle. \end{aligned}$$

Therefore,

$$\langle M_{8}^{in} | \mathcal{X}_{tot}(0) | M_{8}^{in} \rangle \Rightarrow m_0'^2 \text{Tr}(\overline{M}M) + m_1'^2 (\overline{M}_1^3 M_3^1 + \overline{M}_3^1 M_1^3),$$

with no  $\text{Tr}(\overline{M}_1^3 M_3^1 - \overline{M}_3^1 M_1^3)$  that violates the C invariance condition. It is customary to apply this mass formula for the squares of boson masses (in contrast to the case of fermions). In the approximation of ignoring the second order SU(3) breakings, it should not matter whether we apply the mass formula to  $m$  or  $m^2$  because the difference appears only in the second order of breakings. However, empirically, the Gell-Mann-Okubo mass formula seems to work more nicely in squared mass rather than in linear mass. One way to advocate squared mass is that in Feynman diagram calculations boson self-masses are generated in  $m^2$  instead of  $m$ . But this is not a convincing argument.

$$\begin{aligned} \frac{1}{2} (m_K^2 + m_{\overline{K}}^2) &= m_K^2 = \frac{1}{4} (3 m_{\pi}^2 + m_{\pi}^2), \text{ while } m_K = \frac{1}{4} (3 m_{\pi} + m_{\pi}) \\ 0.246 \text{ GeV}^2 & \quad 0.230 \text{ GeV}^2 \quad 0.496 \text{ GeV} \quad 0.446 \text{ GeV}. \end{aligned}$$

These mass formulas can be put into the form that applies to any representation of particles. Let us suppose that  $\mathcal{X}_{tot}(0)$  is sandwiched between N-dimensional representations of B or M. We should construct the 8th component of octet from N-dimensional representation matrices of  $\Lambda_a (a=1,2,\dots,8)$ . There are two ways to construct it:

$$\langle \overline{B(N)}(M(N))^{in} | \mathcal{N}_{tot}(0) | B(N)(M(N))^{in} \rangle = m_0 + m_1 \Lambda_8 + m_2 \sum_{ab} d_{8ab} \Lambda_a \Lambda_b .$$

The third term in the right-hand side can be rewritten as

$$\begin{aligned} d_{8ab} \Lambda_a \Lambda_b &= \sqrt{\frac{1}{3}} (\Lambda_1 \Lambda_1 + \Lambda_2 \Lambda_2 + \Lambda_3 \Lambda_3) - \sqrt{\frac{1}{12}} (\Lambda_4 \Lambda_4 + \Lambda_5 \Lambda_5 + \Lambda_6 \Lambda_6 + \Lambda_7 \Lambda_7) - \sqrt{\frac{1}{3}} \Lambda_8 \Lambda_8 \\ &= \frac{-1}{\sqrt{12}} \sum_a \Lambda_a \Lambda_a + \frac{\sqrt{3}}{2} (\Lambda_1 \Lambda_1 + \Lambda_2 \Lambda_2 + \Lambda_3 \Lambda_3) - \frac{1}{2\sqrt{3}} \Lambda_8 \Lambda_8 \\ &= -\sqrt{\frac{1}{12}} \sum_a \Lambda_a \Lambda_a + 2\sqrt{3} \vec{I}^2 - \frac{\sqrt{3}}{2} Y^2 , \end{aligned}$$

where  $\vec{I}$  and  $Y$  are the  $(N \times N)$  representations of the isospin and the hypercharge operators. In its final form, the Gell-Mann-Okubo mass formula is written in the form of

$$\langle \overline{B(N)}(M(N))^{in} | \mathcal{N}_{tot}(0) | B(N)(M(N))^{in} \rangle = M_0 + M_1 Y + M_2 [I(I+1) - \frac{1}{4} Y^2] .$$

### SU(3) breaking in coupling constants:

The method of deriving formulas is very similar to that of mass formulas. For coupling, for instance, make a tensor that transforms like  $\lambda_8$  out of  $\overline{B(8)}$ ,  $B(8)$ , and  $M(8)$ ;

$$\begin{aligned} &g_1 \text{Tr}(\overline{B} \lambda_8 B) + g_2 \text{Tr}(\overline{B} \lambda_8 M) + g_3 \text{Tr}(\overline{B} \lambda_8 B) + g_4 \text{Tr}(\overline{B} \lambda_8 M B) + g_5 \text{Tr}(\overline{B} M \lambda_8 B) \\ &+ g_6 \text{Tr}(\overline{B} \lambda_8 B M) + g_7 \text{Tr}(\overline{B} M) \text{Tr}(B \lambda_8) + g_8 \text{Tr}(\overline{B} \lambda_8) \text{Tr}(B M) + g_9 \text{Tr}(\overline{B} B) \text{Tr}(M \lambda_8) . \end{aligned}$$

Just as in the  $M(8) + B(8) \rightarrow M(8) + B(8)$  scattering, eight out of nine constants are independent. Furthermore, C invariance imposes constraints for the  $\frac{1}{2} - \frac{1}{2} - 0^+$  couplings as

$$g_1 = g_2, \quad g_3 = g_4, \quad \text{and} \quad g_7 = g_8 .$$

For the  $\overline{B(10)} B(8) M(8)$  couplings, there are 4 independent SU(3) breaking couplings to the lowest order since

$$\begin{aligned} 8 \times 8 (= \overline{B(8)} \times H_{\lambda_8}) &= \underline{1} + \underline{8}_A + \underline{8}_S + \underline{10} + \overline{10} + \underline{27} , \\ 8 \times 10 (= \overline{M(8)} \times B(10)) &= \underline{8} + \underline{10} + \underline{27} + \underline{35} . \end{aligned}$$

There is one testable relation known for the 10-8-8 couplings:

$$\frac{1}{\sqrt{3}} g(\Delta^{++} \rightarrow \pi^+ p) + \sqrt{2} g(\Xi^- \rightarrow \pi^0 \Xi^-) = -\frac{1}{\sqrt{2}} g(\Sigma^{*+} \rightarrow \pi^+ \Sigma^+) + \sqrt{\frac{3}{2}} g(\Xi^{*+} \rightarrow \pi^+ \Lambda) ,$$

where the signs of the coupling constants above depend on your sign /phase conventions of fields.

### Electromagnetic breaking of SU(3) symmetry:

The electromagnetic current transforms under SU(3) like

$$\begin{aligned} J_\mu^{em}(x) &= e \left( \frac{2}{3} \overline{u} \gamma_\mu u - \frac{1}{3} \overline{d} \gamma_\mu d - \frac{1}{3} \overline{s} \gamma_\mu s \right) = e \overline{q} \gamma_\mu \left( \frac{1}{2} \lambda_3 + \frac{1}{2\sqrt{3}} \lambda_8 \right) q \\ &\sim \frac{1}{3} (2T_1^1 - T_2^2 - T_3^3) \end{aligned}$$

The matrix element of  $J_\mu^{em}(x)$  is given in the Lorentz space in the form of

$$\langle \overline{B(8)}(p')^{in} | J_\mu^{em}(0) | B(8)(p)^{in} \rangle = \sqrt{\frac{m^2}{E_p E_{p'}}} \overline{u}_{p's} [\gamma_\mu F_1(q^2) + i \sigma_{\mu\nu} q^\nu F_2(q^2)] u_{ps} \quad \text{with } q=p'-p .$$

$F_1(0)$  is the electric charge and  $F_2(0)$  is the anomalous magnetic moment. Each of  $F_1(q^2)$

and  $F_2(q^2)$  has the following SU(3) structure;

$$\underline{a} [\text{Tr}(\bar{B} \lambda_Q B) - \text{Tr}(\bar{B} B \lambda_Q)] + \underline{b} [\text{Tr}(\bar{B} \lambda_Q B) + \text{Tr}(\bar{B} B \lambda_Q)] \quad \text{with } \lambda_Q \equiv \frac{1}{2} \lambda_3 + \frac{1}{2\sqrt{3}} \lambda_8.$$

However, the  $\underline{b}$  term must be zero for  $F_1(q^2)$  at  $q^2 = 0$ .

$\therefore$  At  $q^2 = 0$  (equal to  $q_\mu = 0$  in the limit of degenerate mass),  $F_1(0)$  must give the electric charge  $Q$ .  $Q$  flips sign under  $B_j^i \rightarrow B_i^j$  and  $\bar{B}_j^i \rightarrow \bar{B}_i^j$ . In the above expression, the  $\underline{a}$  term flips sign under this operation, while the  $\underline{b}$  term does not. Therefore, the  $\underline{b}$  term is zero at  $q^2 = 0$  (not at  $q^2 \neq 0$ ).

On the other hand, the magnetic form factor term  $F_2(q^2)$  has no constraint. Writing the anomalous magnetic moments in terms of  $\underline{a}$  and  $\underline{b}$ , we find

$$\begin{aligned} \mu_p &= a + \frac{1}{3} b, & \mu_n &= -\frac{2}{3} b, & \mu_\Lambda &= -\frac{1}{3} b, & \mu_{\Sigma^+} &= a + \frac{1}{3} b, & \mu_{\Sigma^0} &= \frac{1}{3} b, \\ \mu_{\Sigma^-} &= -a + \frac{1}{3} b, & \mu_{\Xi^0} &= -\frac{2}{3} b, & \mu_{\Xi^-} &= -a + \frac{1}{3} b, & \mu_{\Lambda \Sigma^0} &= \frac{1}{3} b. \end{aligned}$$

Experimentally,

$$\begin{aligned} \mu_p &= 2.792845 - 1, & \mu_n &= -1.91304, & \mu_\Lambda &= -0.613, & \mu_{\Sigma^+} &= 2.38 - 1, & \mu_{\Sigma^-} &= -1.14 + 1, \\ \mu_{\Xi^0} &= -1.250, & \mu_{\Xi^-} &= -0.69 + 1 \quad \text{in the unit of } e\hbar/2m_p c. \end{aligned}$$

The transition magnetic moment  $\mu_{\Lambda \Sigma^0}$  is defined through

$$\langle \Lambda(p')^{\text{in}} | J_\mu^{\text{em}}(0) | \Sigma^0(p)^{\text{in}} \rangle = \quad (\text{the same expression as in the previous page}),$$

and it can be determined from the  $\Sigma^0 \rightarrow \gamma \Lambda$  decay rate. These relations are valid to the first order in the electromagnetic interaction and to all orders of SU(3) symmetric strong interactions. No SU(3) breaking strong interaction ( $\sim \lambda_8$ ) is included. When we test the predictions with experiment, we may compare the anomalous magnetic moments measured in the unit of the common nuclear magneton,  $e\hbar/2m_p c$ , or the unit of  $e\hbar/2m_i c$  with  $m_i$  being  $N, \Lambda, \Sigma, \Xi$ . Since no strong SU(3) breaking is included, we can not tell what is the correct way to compare the predictions with experiment.

Note that the SU(3) predicts

$$\mu_p = \mu_{\Sigma^+}, \quad \mu_n = \mu_{\Xi^0}, \quad \text{and } \mu_{\Sigma^-} = \mu_{\Xi^-}.$$

These relation result because the electromagnetic SU(3) breaking ( $\sim \lambda_Q$ ) commute with the "U spin subgroup";  $[\lambda_Q, \lambda_6] = [\lambda_Q, \lambda_7] = [\lambda_Q, -\frac{1}{2} \lambda_3 + \frac{\sqrt{3}}{2} \lambda_8] = 0$ . Since  $\lambda_Q$  is a U-spin singlet. If one classifies octet components in the U-spin, one find that

$$\begin{aligned} (\Sigma^+, p) &- \text{doublet}, \\ (\Sigma^-, \Xi^-) &- \text{doublet}, \\ (\Xi^0, -\frac{1}{2} \Sigma^0 + \frac{\sqrt{3}}{2} \Lambda, n) &- \text{triplet}. \end{aligned}$$

The expectation value of a U-spin singlet operator is common to all components within the same U-spin multiplet.

The electric charge radius, defined by  $\langle r^2 \rangle = \frac{1}{6} \left( dF_1(q^2)/dq^2 \right) \Big|_{q^2=0}$  obeys the same SU(3) relations as the anomalous magnetic moments.

Electromagnetic mass differences:

This is the second order effect of  $J_\mu^{\text{em}}(x)$  since

$$m_{em} \propto \frac{1}{2} e^2 \int d^4x d^4y \Delta^{\mu\nu}(x-y) \langle \bar{B}_{(g)}^{in} | T(J_\mu^{em}(x) J_\nu^{em}(y)) | B_{(g)}^{in} \rangle$$

The SU(3) structure is  $\lambda_Q \times \lambda_Q = (T_1^1 - \frac{1}{3} T_1^1)(T_1^1 - \frac{1}{3} T_1^1) \sim 1 + T_1^1 + T_{11}^{11}$ .

For B(8), therefore,

$$\Delta m = m_1 \bar{B}_1^1 B_1^1 + m_2 \bar{B}_1^1 B_1^1 + m_3 \bar{B}_1^1 B_1^1 + (\text{nonelectromagnetic terms})$$

$$= m_1' \text{Tr}(\bar{B} \lambda_Q B \lambda_Q) + m_2' \text{Tr}(\bar{B} \lambda_Q B) + m_3' \text{Tr}(\bar{B} B \lambda_Q) + (\text{non e.m. terms}). [\lambda_Q^2 = c_1 1 + c_2 \lambda_8]$$

Explicitly,  $+ m_1'' \text{Tr}(\bar{B} \lambda_Q) \text{Tr}(B \lambda_Q)$  (but not independent)

$$m_p = m_N + m_3, m_n = m_N, m_{\Sigma^+} = m_{\Sigma} + m_3, m_{\Sigma^0} = m_{\Sigma} + \frac{1}{2}(m_1 + m_2 + m_3), m_{\Sigma^-} = m_{\Sigma} + m_2,$$

$$m_{\Xi^0} = m_{\Xi}, m_{\Xi^-} = m_{\Xi} + m_2. \quad \begin{matrix} -1.29 & -6.4 & -7.97 \\ (m_p - m_n) & (m_{\Sigma^0} - m_{\Sigma^-}) & (m_{\Sigma^+} - m_{\Sigma^-}) \end{matrix} \quad [\text{Coleman-Glashow}]$$

## 2.2f Nonets

We often find nine hadrons with the same  $J^P$  and approximately the same masses. Each set consists of one singlet and one octet which are close in mass. Because of the SU(3) breaking transforming like  $\lambda_8$  of octet, a singlet and the  $I=Y=0$  component of  $\underline{8}$  mix with each other and two states of  $I=Y=0$  appear in two linear combinations of  $\underline{1}$  and  $\underline{8}$ . We often call such approximately degenerate  $\underline{1}$  and  $\underline{8}$  as a nonet. (There is no 9 dimensional representation of SU(3).)

$J^P$	$I = 1/2$	$I = 1$	$I = 0$
$1^-$	$K^*, \bar{K}^* (890 \text{ MeV})$	$\rho (770 \text{ MeV})$	$\omega (780 \text{ MeV}) \quad \phi (1020 \text{ MeV})$
$2^+$	$K^*, \bar{K}^* (1420 \text{ MeV})$	$A_2 (1310 \text{ MeV})$	$f (1260 \text{ MeV}) \quad f' (1514 \text{ MeV})$
$3/2^-$	$\begin{pmatrix} N' (1520 \text{ MeV}), \\ \Xi' (1820 \text{ MeV}), \end{pmatrix}$	$\Sigma' (1670 \text{ MeV})$	$\Lambda' (1520 \text{ MeV}), \quad \Lambda' (1690 \text{ MeV})$

Take the case of  $J^P = 1^-$ . SU(3) predicts the following:

(a) G-M-O mass formula.  $\frac{1}{2}(m^2(K^*) + m^2(\bar{K}^*)) = \frac{1}{4}(3 m^2(\phi_8) + m^2(\rho))$ .

Here,  $\phi_8 = (\bar{u}u + \bar{d}d - 2\bar{s}s)/\sqrt{6}$ . Plugging in the experimental values, we find

$$\frac{1}{2}(m^2(K^*) + m^2(\bar{K}^*)) = m^2(K^*) = 0.795 \text{ GeV}^2,$$

$$\text{while } \left. \begin{aligned} \frac{1}{4}(3 m_\omega^2 + m_\rho^2) &= \frac{1}{4}(3 \times 0.614 + 0.593) = 0.609 \text{ GeV}^2 \\ \frac{1}{4}(3 m_\phi^2 + m_\rho^2) &= \frac{1}{4}(3 \times 1.040 + 0.593) = 0.928 \text{ GeV}^2 \end{aligned} \right\} \neq 0.795 \text{ GeV}^2.$$

(b)  $1^- \rightarrow 0^- + 0^-$  decay couplings. Note first that  $\underline{1} \not\rightarrow \underline{8} + \underline{8}$  because the singlet made of two  $\underline{8}$  is symmetric under interchange of two  $\underline{8}$ , while the  $1^- \rightarrow 0^- \rightarrow 0^-$  coupling has to be necessarily antisymmetric,  $(\phi_a \partial_\mu \phi_b - \partial_\mu \phi_a \phi_b) V_c^\mu$  (antisymmetric in  $a \leftrightarrow b$ ). Then keep in mind that small SU(3) breakings in mass may cause large deviation from SU(3) symmetric prediction unless one separate SU(3) breaking effects as much as possible. In the decay rate of p-wave,  $\Gamma \sim g_{VPP}^2 p^3/m_V^2$ , we should define the reduced width  $\bar{\Gamma} \equiv \Gamma/p^3$  (or  $\bar{\Gamma} \equiv \Gamma \times m_V^2/p^3$ ) and compare  $\bar{\Gamma}$  with the SU(3) predictions. [In case of  $\ell$ -th wave decay, the reduced width should be  $\Gamma/p^{2\ell+1}$ .] The SU(3) symmetry for the  $1^- \rightarrow 0^- \rightarrow 0^-$  coupling ( $\underline{8}_A$  only) predicts

$$\bar{\Gamma}(\rho^+ \rightarrow \pi^+ \pi^0) : \bar{\Gamma}(K^{*+} \rightarrow K^0 \pi^+) : \bar{\Gamma}(\phi_8 \rightarrow K^+ K^-) = 1 : \frac{1}{2} : \frac{3}{4}.$$

Experimentally from the observed decay widths,

$$\bar{\Gamma}(\rho^+ \rightarrow \pi^+ \pi^0) : \bar{\Gamma}(K^{*+} \rightarrow K^0 \pi^+) : \bar{\Gamma}(\phi \rightarrow K^+ K^-) = 1.86 : 1.10 : 1.00.$$

(c)  $1^- \rightarrow e^+ e^-$  decay rates. Since  $J_\mu^{em}$  transforms like sum of  $\lambda_3$  and  $\frac{\lambda_8}{\sqrt{3}}$  of  $\underline{8}$ , only  $\rho^0$  and  $\phi_8$  can annihilate into  $e^+ e^-$  through the electromagnetic interaction of  $O(e)$ .

The transition matrix elements are written in the form of

$$V(8) \rightarrow \begin{array}{c} e^- \\ \gamma \\ e^+ \end{array} \sim \sqrt{\frac{m_e^2}{E_k E_{k'}}} (\bar{u}_{ks} \gamma_\mu v_{k's'}) \frac{-ig \mu \gamma}{(k+k')^2} \langle 0^{in} | J_\mu^{em}(0) | V(8) \rangle.$$

From the fact that  $J_\mu^{em} \sim \frac{\lambda_3}{2} + \frac{\lambda_8}{2\sqrt{3}}$ , we find that

$$\langle 0^{in} | J_\mu^{em}(0) | \rho^0 \rangle : \langle 0^{in} | J_\mu^{em}(0) | \phi_8 \rangle : \langle 0^{in} | J_\mu^{em}(0) | \omega_1 \rangle = 1 : \frac{1}{\sqrt{3}} : 0.$$

These decays are s-wave decays, so the reduced widths are  $\bar{\Gamma}/p \propto |\langle 0^{in} | J_\mu^{em}(0) | V \rangle|^2$ . From the experimentally observed widths,

$$\bar{\Gamma}(\rho^0 \rightarrow e^+ e^-) : \bar{\Gamma}(\omega \rightarrow e^+ e^-) : \bar{\Gamma}(\phi \rightarrow e^+ e^-) = [18.0 : 1.92 : 2.64] \times 10^{-6}.$$

All of (a), (b), and (c) show that neither of  $\omega$  and  $\phi$  really fits in  $\phi_8$  nor  $\omega_1$ . One might argue that deviations from SU(3) symmetry limit are due to large SU(3) breaking effects. But the breaking effects appear to be much larger than anywhere else. Why? The reason is that  $\phi_8$  and  $\omega_1$  are nearly degenerate in mass and even a small SU(3) breaking generates a large mixing of the two states (degenerate perturbation). Under this circumstance, we can hope that SU(3) predictions are applicable after the mixing is taken into account. Let us write the  $2 \times 2$  mass matrix in the  $\phi_8$ - $\omega_1$  or  $\phi$ - $\omega$  space:

$$m^2 + \delta m^2 = \begin{pmatrix} m_8^2 + \langle \phi_8^{in} | \mathcal{H}_{br} | \phi_8^{in} \rangle & \langle \phi_8^{in} | \mathcal{H}_{br} | \omega_1^{in} \rangle \\ \langle \omega_1^{in} | \mathcal{H}_{br} | \phi_8^{in} \rangle & m_1^2 \end{pmatrix},$$

$$= \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} m_8^2 & 0 \\ 0 & m_1^2 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

Here  $m_1$  and  $m_8$  are the masses of  $1$  and  $8$  in the SU(3) symmetric limit, and  $\mathcal{H}_{br}$  is the SU(3) breaking of  $\lambda_8$  of  $8$ . The eigenstates of mass are

$$\begin{cases} \phi = \phi_8 \cos \theta - \omega_1 \sin \theta, \\ \omega = \phi_8 \sin \theta + \omega_1 \cos \theta, \end{cases}$$

with

$$\tan 2\theta = \frac{\langle \omega_1^{in} | \mathcal{H}_{br} | \phi_8^{in} \rangle + \langle \phi_8^{in} | \mathcal{H}_{br} | \omega_1^{in} \rangle}{m_1^2 - m_8^2 - \langle \phi_8^{in} | \mathcal{H}_{br} | \phi_8^{in} \rangle}.$$

The mixing angle  $\theta$  is large, as is expected. It can be determined from experiment as follows:

$$(a) \text{ From the mass formulas. } m^2(\rho) = m_8^2 + \langle \rho^{in} | \mathcal{H}_{br} | \rho^{in} \rangle, \quad (1)$$

$$m^2(K^*) = m_8^2 + \langle K^{*in} | \mathcal{H}_{br} | K^{*in} \rangle, \quad (2)$$

$$m^2(\omega) + m^2(\phi) = m_1^2 + m_8^2 + \langle \phi_8^{in} | \mathcal{H}_{br} | \phi_8^{in} \rangle,$$

$$\text{from } \begin{cases} \phi_8 = \phi \cos \theta + \omega \sin \theta, \\ \omega_1 = -\phi \sin \theta + \omega \cos \theta, \end{cases} \text{ and } \begin{cases} m_8^2 + \langle \phi_8^{in} | \mathcal{H}_{br} | \phi_8^{in} \rangle = m_\phi^2 \cos^2 \theta + m_\omega^2 \sin^2 \theta, \\ m_1^2 = m_\phi^2 \sin^2 \theta + m_\omega^2 \cos^2 \theta. \end{cases} \quad (3)$$

The SU(3) symmetry requires ( $8_S$  only)

$$\langle \rho^{in} | \mathcal{H}_{br} | \rho^{in} \rangle : \langle K^{*in} | \mathcal{H}_{br} | K^{*in} \rangle : \langle \phi_8^{in} | \mathcal{H}_{br} | \phi_8^{in} \rangle = -2 : 1 : 2.$$

The four unknown parameters ( $m_1^2$ ,  $m_8^2$ ,  $\langle \phi_8^{in} | \mathcal{H}_{br} | \phi_8^{in} \rangle$ ,  $\theta$ ) with the four constraints, (1), (2), (3), & (4).

We obtain  $\theta = 41^\circ$  or  $-41^\circ$  (fairly sensitive to small errors in mass values).

(b) From  $1^- \rightarrow 0^- 0^-$  decays.

SU(3) predicts  $\bar{\Gamma}(\phi \rightarrow K^+ K^-) = \frac{3}{4} \cos^2 \theta \bar{\Gamma}(\rho^+ \rightarrow \pi^+ \pi^0)$  instead of  $\bar{\Gamma}(\phi_8 \rightarrow K^+ K^-) = \frac{3}{4} \bar{\Gamma}(\rho^+ \rightarrow \pi^+ \pi^0)$ . This relation leads us to  $\cos^2 \theta = 0.717$  or  $\theta = \pm 29^\circ$ .

(c) From  $1^- \rightarrow e^- e^+$  decays.

SU(3) predicts  $\bar{\Gamma}(\rho^0 \rightarrow e^+ e^-) : \bar{\Gamma}(\phi \rightarrow e^+ e^-) : \bar{\Gamma}(\omega \rightarrow e^+ e^-) = 1 : \frac{1}{3} \cos^2 \theta : \frac{1}{3} \sin^2 \theta$ .

Using the experimental data on  $\bar{\Gamma}(\rho^0 \rightarrow e^+ e^-) / \bar{\Gamma}(\omega \rightarrow e^+ e^-)$ , we obtain  $\theta = \pm 32^\circ$ . Using the experimental data on  $\bar{\Gamma}(\omega \rightarrow e^+ e^-) / \bar{\Gamma}(\phi \rightarrow e^+ e^-)$ , we obtain  $\theta = \pm 38^\circ$ . They indicate typical errors involved in this kind of estimates.

All of the experimental observations (a), (b), and (c) point to the value  $\theta \approx 30^\circ$ . In order to determine the sign of  $\theta$ , we need a little theoretical consideration. We favor  $\theta \approx +30^\circ$  rather than  $-30^\circ$  leading to  $\phi$  consisting dominantly of  $\bar{s}s$  or  $T_3^3$ . For  $\tan \theta = \sqrt{1/2}$  ( $\theta \approx 35^\circ$ ), (called the ideal mixing)

$$\phi = \frac{1}{\sqrt{6}} (\bar{u}u + \bar{d}d - 2\bar{s}s) \cos \theta - \frac{1}{\sqrt{3}} (\bar{u}u + \bar{d}d + \bar{s}s) \sin \theta = -\bar{s}s,$$

$$\omega = \frac{1}{\sqrt{6}} (\bar{u}u + \bar{d}d - 2\bar{s}s) \sin \theta + \frac{1}{\sqrt{3}} (\bar{u}u + \bar{d}d + \bar{s}s) \cos \theta = \frac{1}{\sqrt{2}} (\bar{u}u + \bar{d}d)$$

For  $\tan \theta = -\sqrt{1/2}$ ,

$$\phi = (2\bar{u}u + 2\bar{d}d - \bar{s}s)/3 \quad \text{and} \quad \omega = \frac{1}{3\sqrt{2}} (\bar{u}u + \bar{d}d + 4\bar{s}s).$$

The reason why we favor  $\phi \sim -\bar{s}s$  is based on the empirical selection rule called the Okubo-Zweig-Iizuka rule (or the OZI rule). The OZI rule requires that hadronic processes, either decays or scatterings, are suppressed substantially when they involve pair annihilation/creation of  $s$  and  $\bar{s}$ . If we apply this rule to the  $\phi$  decay, we come to the conclusion that

if  $\phi \sim -\bar{s}s$ , the decays of  $\phi$  into  $\pi^+ \pi^- \pi^0$ ,  $\rho^+ \pi^-$ ,  $\rho^0 \pi^+$ , etc are suppressed, while  $\phi \rightarrow K\bar{K}$  is not.

Experimentally, the  $\phi \rightarrow K\bar{K}$  is the main decay mode of  $\phi$  despite the tiny phase space (1020 MeV -  $2 \times 497$  MeV in p-wave in contrast to  $\phi \rightarrow 3\pi$  with 1020 MeV -  $3 \times 140$  MeV). Another supporting evidence for the OZI rule is found in  $\phi$  production processes;

$$\frac{\sigma(\pi^- p \rightarrow \phi n)}{\sigma(\pi^- p \rightarrow \omega n) \text{ or } \sigma(\pi^- p \rightarrow \rho^0 n)} = \frac{1}{50} \sim \frac{1}{100}.$$

Therefore, we are now confident with the assignment that  $\phi \sim -\bar{s}s$ . The OZI rule was rather a mirky empirical rule in 1960's to early 1970's. In 1974, this rule suddenly received a dramatic experimental verification through the discovery of  $\psi$  particle ( $c\bar{c}$  bound state). The justification based on the asymptotic freedom of QCD was also given.

#### Nonet coupling hypothesis (Okubo)

Group theoretically, the coupling of  $V(1)$  and the coupling of  $V(8)$  are not related to each other in any way; they are two independent couplings in group theory. However, there was a theoretical conjecture as early as in early 1960's that the coupling of  $V(1)$  may be related to the coupling of  $V(8)$  in a simple manner. Such relations have been verified experimentally thereafter. They are indications that we may be able to

learn a lot of physics by building a physical or dynamical (as opposed to group theoretical) model of hadrons based on the quark picture. The nonet coupling assumption is stated for the  $1^- - \frac{1}{2}^+ - \frac{1}{2}^+$  couplings as an example in the following way:

SU(3) group theory only:  $g_1 \text{Tr}(\bar{B}_8 V_8 B_8) + g_2 \text{Tr}(\bar{B}_8 B_8 V_8) + g_3 \text{Tr}(\bar{B}_8 B_8) \cdot V_1$ .

Nonet coupling hypothesis:  $g_1 \text{Tr}(\bar{B}_8 V_{8-1} B_8) + g_2 \text{Tr}(\bar{B}_8 B_8 V_{8-1})$ ,

where

$$V_{8-1} = \begin{pmatrix} \rho^0/\sqrt{2} + \omega/\sqrt{2}, & \rho^+, & K^{*+} \\ \rho^-, & -\rho^0/\sqrt{2} + \omega/\sqrt{2}, & K^{*0} \\ K^{*-}, & \bar{K}^{*0}, & -\phi \end{pmatrix} = (V_8)_j^i + \frac{1}{\sqrt{3}} \delta_j^i V_1.$$

The similar assignment for the  $2^+$  mesons ( $A_1, K^*, f, f'$ ).

The ideal mixing is not realized for the  $Q^-$  mesons ( $\pi, K, \bar{K}, \eta, \eta'$ ). ( $\pi, K, \bar{K}, \eta$ ) are approximately an octet and  $\eta'$  is approximately a singlet. By more detailed analysis, one can determine a small mixing between  $\eta$  and  $\eta'$  with mixing angle  $\theta$ . The angle  $\theta$  can be determined from many decay modes such as  $1^- \rightarrow 0^- + \gamma$  as well as from the deviation of the masses from the G-M-O formula. From the G-M-O deviation,  $\theta = \pm 10^\circ$  or so. From the decay mode analysis,  $\theta \approx -10^\circ$  is favored.

### 2.3 Static quark model and SU(6) spin-unitary spin symmetry.

#### 2.3a. SU(6) transformations of spin and unitary spin

SU(3) transformations  $\exp(i\frac{1}{2}\lambda_a \alpha_a)$  do not change spin directions (nor any other space-time property) of quarks. SU(2) spin rotations  $\exp(i\frac{1}{2}\sigma_j \theta_j)$  do not change flavors (u,d,s) of quarks. The strong interaction Hamiltonian seems to be approximately invariant under the SU(3) transformation and, if quarks are nearly at rest, invariant under spin rotations, too. [If quarks are relativistic, the Hamiltonian is invariant only under the simultaneous rotation of spin and orbital parts, namely the rotation by  $\vec{J}$ .] Provide that the quarks inside of hadrons are approximately at rest, can we expect that the strong interactions are approximately invariant under combined transformations of SU(3) flavors and SU(2) spin? The answer is Yes.

Since we consider transformations which mix not only flavors but also spins at the same time, we are actually dealing with the most general transformations among the six objects, u-quark spin up and down, d-quark spin up and down, and s-quark spin up and down. The transformations are SU(6) group transformations. Written in the exponentiated form, the rotations consist of

$$\exp(i\frac{1}{2}\lambda_a \alpha_a), \quad \exp(i\frac{1}{2}\sigma_j \theta_j), \quad \text{and} \quad \exp(i\frac{1}{2}\lambda_a \sigma_j \theta_{aj}).$$

The transformation included in the third one above is, for instance,

$$\exp(i\frac{1}{2}\lambda_1 \sigma_2 \theta) \approx 1 + \frac{1}{2}i\theta \lambda_1 \sigma_2 + O(\theta^2),$$

$$\begin{pmatrix} u_\uparrow \\ u_\downarrow \end{pmatrix} \rightarrow \begin{pmatrix} u_\uparrow \\ u_\downarrow \end{pmatrix} + \frac{1}{2}\theta \begin{pmatrix} d_\downarrow \\ -d_\uparrow \end{pmatrix}, \quad \begin{pmatrix} d_\uparrow \\ d_\downarrow \end{pmatrix} \rightarrow \begin{pmatrix} d_\uparrow \\ d_\downarrow \end{pmatrix} + \frac{1}{2}\theta \begin{pmatrix} u_\downarrow \\ -u_\uparrow \end{pmatrix}, \quad \begin{pmatrix} s_\uparrow \\ s_\downarrow \end{pmatrix} \rightarrow \begin{pmatrix} s_\uparrow \\ s_\downarrow \end{pmatrix},$$

and therefore

$$\begin{aligned}
\pi^+ \left( \frac{1}{\sqrt{2}} (\bar{d}_\downarrow u_\uparrow + \bar{d}_\uparrow u_\downarrow) \right) &\rightarrow \pi^+ + \frac{1}{2} \theta \frac{1}{\sqrt{2}} \left( \bar{d}_\downarrow (d_\downarrow) + (\bar{u}_\uparrow) u_\uparrow + \bar{d}_\uparrow (-d_\uparrow) + (-\bar{u}_\downarrow) u_\downarrow \right) \\
&= \pi^+ + \frac{1}{2} \theta \left( \frac{1}{\sqrt{2}} (\bar{u}_\uparrow u_\uparrow - \bar{d}_\uparrow d_\uparrow) - \frac{1}{\sqrt{2}} (\bar{u}_\downarrow u_\downarrow - \bar{d}_\downarrow d_\downarrow) \right) \\
&= \pi^+ + \frac{1}{2} \theta (p^0(s_z=1) - p^0(s_z=-1)).
\end{aligned}$$

Or equivalently,

$$\begin{aligned}
\frac{1}{2\sqrt{2}} \bar{q} \cdot (\lambda_1 - i\lambda_2) q &\rightarrow \frac{1}{2\sqrt{2}} \bar{q} (1 - \frac{1}{2} i\theta \lambda_1 \sigma_2) \cdot (\lambda_1 - i\lambda_2) (1 + \frac{1}{2} i\theta \lambda_1 \sigma_2) q \\
&= \frac{1}{2\sqrt{2}} \bar{q} \cdot (\lambda_1 - i\lambda_2) q + \frac{1}{2\sqrt{2}} i\theta \bar{q} (\lambda_1 \sigma_2 - \lambda_1 \lambda \sigma_2) q \\
(\lambda_- \equiv \lambda_1 - i\lambda_2) &= \frac{1}{2\sqrt{2}} \bar{q} \cdot (\lambda_1 - i\lambda_2) q + \frac{1}{2\sqrt{2}} i\theta \left( \bar{q} \lambda_3 \frac{\sigma_1 - i\sigma_2}{2} q - \bar{q} \lambda_3 \frac{\sigma_1 + i\sigma_2}{2} q \right) \\
&= \pi^+ + \frac{1}{2} \theta (p^0(s_z=1) - p^0(s_z=-1))
\end{aligned}$$

The combined spin-unitary spin rotations change both spin and flavor by one operation. SU(6) group has  $6^2 - 1$  generators [all possible hermitian matrices of  $6 \times 6$  less unit matrix]. When we express them as we did in the previous page, they obey the commutation relations as follows:

$$[\frac{1}{2} \lambda_a, \frac{1}{2} \lambda_b] = i f_{abc} \frac{1}{2} \lambda_c \quad (a, b, c = 1, 2, 3 \dots 8) \quad \text{SU(3) subgroup of SU(6).}$$

$$[\frac{1}{2} \sigma_i, \frac{1}{2} \sigma_j] = i \epsilon_{ijk} \frac{1}{2} \sigma_k \quad (i, j, k = 1, 2, 3) \quad \text{SU(2) subgroup of SU(6).}$$

$$[\frac{1}{2} \sigma_i, \frac{1}{2} \lambda_a] = 0,$$

$$[\frac{1}{2} \sigma_i \lambda_a, \frac{1}{2} \lambda_b] = i f_{abc} \frac{1}{2} \sigma_i \lambda_c,$$

$$[\frac{1}{2} \sigma_i \lambda_a, \frac{1}{2} \sigma_j] = i \epsilon_{ijk} \frac{1}{2} \sigma_k \lambda_a,$$

$$\begin{aligned}
[\frac{1}{2} \sigma_i \lambda_a, \frac{1}{2} \sigma_j \lambda_b] &= 2 [\frac{1}{2} \sigma_i, \frac{1}{2} \sigma_j] \left\{ \frac{1}{2} \lambda_a, \frac{1}{2} \lambda_b \right\}_+ + 2 \left\{ \frac{1}{2} \sigma_i, \frac{1}{2} \sigma_j \right\}_+ [\frac{1}{2} \lambda_a, \frac{1}{2} \lambda_b] \\
&= i \epsilon_{ijk} d_{abc} \frac{1}{2} \sigma_k \lambda_c + i \delta_{ij} f_{abc} \frac{1}{2} \lambda_c.
\end{aligned}$$

Here  $\underline{c}$  is over  $\underline{0}, 1, 2 \dots 8$ , with  $\lambda_0 \equiv \sqrt{2/3} \cdot 1$ .

The algebra closes with these rotations.

### 2.3b. Particle assignment

Irreducible representations of SU(6) are constructed from products of  $\underline{6}$  (and  $\bar{\underline{6}}$ , if you wish to use it), just as we did for SU(3). However, from the physical understanding, it is often more convenient to express states in term of a pair of indices ( $i, a$ ) referring to SU(2) spins and SU(3) flavors instead of a single index running over 1 to 6.

Mesons (from  $\bar{q}q$ )

$$\underline{1} \text{ (singlet)} \quad \bar{q}^{a,i} q_{a,i} / \sqrt{6} = (\bar{u}_\downarrow u_\uparrow + \bar{u}_\uparrow u_\downarrow + \bar{d}_\downarrow d_\uparrow + \bar{d}_\uparrow d_\downarrow + \bar{s}_\downarrow s_\uparrow + \bar{s}_\uparrow s_\downarrow) / \sqrt{6}.$$

35 (adjoint representation like  $\underline{8}$  of SU(3))

$$\begin{aligned}
\bar{q}^{a,i} q_{b,j} - \frac{1}{6} \delta_b^a \delta_j^i \bar{q}^{c,k} q_{c,k} \\
= \frac{1}{3} \delta_b^a (\bar{q}^{c,i} q_{c,j} - \frac{1}{2} \delta_j^i \bar{q}^{c,k} q_{c,k}) + (\text{continued to the next page}) \\
\sim \underline{(1, 3)}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \delta_j^i (\bar{q}^{a,k} q_{b,k} - \frac{1}{3} \delta_b^a \bar{q}^{c,k} q_{c,k}) \dots\dots (8, 1) \\
& + \left( \bar{q}^{a,i} q_{b,j} - \frac{1}{3} \delta_b^a \bar{q}^{c,i} q_{c,j} - \frac{1}{2} \delta_j^i \bar{q}^{a,k} q_{b,k} + \frac{1}{6} \delta_b^a \delta_j^i \bar{q}^{c,k} q_{c,k} \right) \\
& \dots\dots (8, 3)
\end{aligned}$$

Namely,  $\underline{6} \times \underline{6} = \underline{1} + \underline{35}$ ,  $\underline{1}$   $\underline{35}$

$$(\underline{3}, \underline{2}) \times (\underline{3}, \underline{2}) = (\underline{8} + \underline{1}, \underline{3} + \underline{1}) = (\underline{1}, \underline{1}) + ((\underline{1}, \underline{3}) + (\underline{8}, \underline{1}) + (\underline{8}, \underline{3}))$$

The SU(6) group does not affect the orbital part of states, so particles assigned to the same representation must have the same orbital angular momentum. If we consider the s-wave bound states of  $\bar{q}q$ , we can assign  $^1S_1$  and  $^3S_1$  states in  $\underline{35}$  as follows:

$$\begin{aligned}
\underline{35} &= (\underline{1}, \underline{3}) + (\underline{8}, \underline{1}) + (\underline{8}, \underline{3}) \\
&\quad \omega_1 \quad (\pi, K, \bar{K}, \eta) \quad (\rho, K^*, \bar{K}^*, \phi)
\end{aligned}$$

The SU(3) singlet (apart from a small mixing to  $\bar{8}$ )  $\eta'$  meson does not enter  $\underline{35}$ . It should belong to SU(6) singlet. This particle assignment tells us about the degree of approximation of SU(6); we consider the limit where  $\pi, K, \eta, \rho, \omega$ , and  $\phi$  are all degenerate. In spite of such approximation, the SU(6) classification of hadrons and SU(6) predictions of some of the coupling constants work remarkably well.

Baryons Triple products of  $q$ . The lowest states are presumably those entirely in s-wave.

$$\begin{aligned}
q_{a,i} q_{b,j} q_{c,k} &= \begin{array}{|c|c|c|} \hline a,i & b,j & c,k \\ \hline \end{array} + \begin{array}{|c|c|} \hline a,i & b,j \\ \hline c,k & \end{array} \begin{array}{|c|c|} \hline a,i & c,k \\ \hline b,j & \end{array} + \begin{array}{|c|} \hline a,i \\ \hline b,j \\ \hline c,k \\ \hline \end{array} \\
\underline{6} \times \underline{6} \times \underline{6} &= \underline{56} + \underline{70} + \underline{70} + \underline{20} \\
\text{SU(3) x SU(2)} & \quad (\underline{10}, \underline{4}) \quad (\underline{10}, \underline{2}) \quad (\underline{8}, \underline{4}) \quad (\underline{8}, \underline{2}) \\
\text{content} & \quad \begin{array}{|c|c|c|} \hline a & b & c \\ \hline \end{array} \times \begin{array}{|c|c|c|} \hline i & j & k \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \times \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \end{array} \\
& \quad (\underline{8}, \underline{2}) \quad (\underline{8}, \underline{2}) \quad (\underline{1}, \underline{2}) \quad (\underline{1}, \underline{4}) \\
& \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \times \begin{array}{|c|} \hline \square \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \square & \square \\ \hline \end{array} \times \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \end{array}
\end{aligned}$$

where the product of Young tableaux in SU(3) and SU(2) spaces are so combined as to make a specified overall symmetry in SU(6)

The baryons,  $N, \Lambda, \Sigma, \Xi$  belong to  $(\underline{8}, \underline{2})$  and the  $J^P = 3/2^+$  resonances,  $\Delta, \Sigma', \Xi', \Omega$  belong to  $(\underline{10}, \underline{4})$ . Therefore,  $\underline{56}$  of SU(6) nicely accommodates the octet baryons of spin 1/2 and the decuplet of baryon resonances of spin 3/2 (they have the same parity). The other representations,  $\underline{70}$  and  $\underline{20}$ , also group together nicely the existing baryon resonances of higher masses.

### 2.3c SU(6) symmetry breaking

Lorentz invariant theory can not satisfy SU(6) symmetry! The free Lagrangian with degenerate mass violate SU(6) symmetry.

$$\mathcal{L}_0 = i \bar{q} \not{\partial} q - m \bar{q} q$$

This Lagrangian is invariant if the spinors are rotated by  $\exp(\frac{i}{4}\sigma_{ij}\omega^{ij})$  [ $i,j=1,2,3$ ] and the coordinates are rotated as  $\vec{x} \rightarrow \vec{x}' = R\vec{x}$  ( $R$  is the familiar 3-dimensional rotation matrix). Only if we consider the quark and antiquark at rest,

$$\mathcal{L}_0 = i \bar{q} \gamma_0 \frac{\partial}{\partial t} q - m \bar{q} q$$

is invariant under  $\exp(\frac{i}{4}\sigma_{ij}\omega^{ij})$  alone. If we write such Lagrangian for each flavor and add them up, we obtain a free Lagrangian which is invariant not only under  $SU(3)$   $SU(2)$ , but also under  $SU(6)$ . Note that

$$(\bar{u}\gamma_0 \frac{\partial}{\partial t} u + \bar{d}\gamma_0 \frac{\partial}{\partial t} d + \bar{s}\gamma_0 \frac{\partial}{\partial t} s) \quad \text{and} \quad -m(\bar{u}u + \bar{d}d + \bar{s}s)$$

are both singlets of  $SU(6)$ .

What is an  $SU(6)$  invariant interaction? Provided that we consider only quarks and antiquarks at rest, we can write several examples. The simplest one is

$$\mathcal{L}_{\text{int}} = f \sum_{i=u,d,s} \bar{q}^i \gamma_\mu q_i V^\mu = f \sum_{i=u,d,s} \bar{q}^i \gamma_0 q_i V^0 = f \sum_{\substack{i=u,d,s \\ n=\text{spins}}} \chi^{+n} \chi_n V^0,$$

where  $V^\mu$  is the  $SU(3)$  singlet vector particle field which will mediate a coulombic force between quarks and antiquarks. Since the fourth component of four-vector is invariant under spin rotations, it is invariant under spin  $SU(2)$ . If we take this model interaction seriously, the force is attractive between  $q$  and  $\bar{q}$  and repulsive between  $q$  and  $q$ , and between  $\bar{q}$  and  $\bar{q}$ . Its implication is that this force can not bind  $qqq$  into baryons. A little more complicated example of  $SU(6)$  invariant interactions is

$$\mathcal{L}_{\text{int}} = g (\bar{q} \frac{1}{2} \lambda_a q \phi_a + \bar{q} \frac{1}{2} \lambda_a \gamma_\mu \gamma_5 q V_a^\mu + \bar{q} \frac{1}{2} \lambda_0 \gamma_\mu \gamma_5 q V^\mu)$$

where  $(\phi_a, V_a^\mu, V^\mu)$  form 35 of  $SU(6)$ , presumably  $J^P = (0^+, 1^+, 1^+)$ . The  $SU(6)$  structure of this interaction is 35  $\times$  35  $\rightarrow$  1.

### 2.3d Color quantum numbers

Notice that the three quarks in 56 are totally symmetric under simultaneous interchange of spin and unitary spin indices. However, we assume that the three quarks in the lowest baryons of  $J^P = 1/2^+$  and the baryon resonances of  $J^P = 3/2^+$  are in s-wave. Then we come to a contradiction with the Fermi statistics because interchange of a pair of quarks in the lowest 56 results in no minus sign to the wave-functions. We can avoid this dilemma by any one of the following three options (maybe more, if you include weird possibilities):

- The three quarks are bound in relative p-wave. This is a clumsy solution. First, we must explain why the p-wave states come out as the lowest states instead of the s-wave states. If the quarks are really in p-wave, we expect more 56 with the same  $P$  and different  $J$  which are nearly degenerate with  $(N, \Sigma, \Xi, \Omega, \Delta, \Sigma', \Xi', \Omega)$ .
- Invent a new statistics that allows particles to occupy the same state up to 3, but no more than 3 (parastatistics).
- Introduce new quantum numbers (called "colors") and assign three colors to each flavor (referring to  $u, d, s, c, b, t$ ) of quark.

In the third option, the three quarks in 56 carry three different colors, so there is no conflict with the ordinary Fermi statistics. For instance,  $\Delta^{++}$  of spin  $S_z = +\frac{3}{2}$  is made of  $u_{\uparrow}$ (red),  $u_{\uparrow}$ (blue), and  $u_{\uparrow}$ (yellow). If the wave-function is totally anti-symmetric in color space,  $\Delta^{++}$  satisfies the required antisymmetry under interchange of any pair of quarks inside.

Furthermore, postulate an unbroken SU(3) symmetry for rotations of three colors. The total antisymmetry in color SU(3) space means that 56 baryons are color singlets. We generalize this reasoning and set up the hypothesis that quarks carry three colors and hadrons are all bound in color singlets.

	SU(6)		color SU(3)
Baryons:	$\square\square\square$	$\times$	$\square$
Mesons:	$\square$ $\square$	$\times$	$\square$ $\square$ } .....(pair of colors contracted)

As long as the hadron spectroscopy is concerned, the option (b), parastatistics, leads us to basically the same conclusions as (c), <sup>but</sup>  $\Delta$  there have been many dramatic dynamical evidences in favor of (c), such as asymptotic freedom. We choose (c) as correct.

Now it is easy to construct an SU(6) symmetric force that is responsible for binding of hadrons: Introduce vector bosons which are 8 of "color" SU(3) [hereafter SU(3)<sub>c</sub>] and 1 of flavor SU(3) [hereafter SU(3)<sub>f</sub>]. The reason why we introduce a color octet of vector particles, instead of a color singlet, is that such vector particles produce through one-particle exchange

$$\begin{aligned} \text{attractive forces in} & \quad \begin{cases} (qq) \text{ in color } \bar{3}, [ \text{ any } (qq) \text{ pair in } \underline{56} \text{ is in color } \bar{3}. ] \\ (\bar{q}q) \text{ in color } \underline{1}, \end{cases} \\ \text{repulsive forces in} & \quad \begin{cases} (qq) \text{ in color } \underline{6}, \\ (\bar{q}q) \text{ in color } \underline{8}. \end{cases} \end{aligned}$$

The signs of the forces provide dynamical justification for the hypothesis that hadrons are bound only in color singlets. The static potential of the one-particle exchange is written in the form of

$$\frac{g^2}{4\pi} \frac{1}{r} \sum_{a=1}^8 (\bar{q}^i (\frac{\lambda_a}{2})^j_i q_j) (\bar{q}^m (\frac{\lambda_a}{2})^n_m q_n)$$

where all indices (i, j, m, n, and a) are those of color SU(3). For the  $\bar{q}q$  potential, we

obtain from  $\sum_a (\lambda_a^j_i (\lambda_a^m_n) = \frac{16}{9} \delta^m_n \delta^j_i - \frac{1}{3} \sum_a (\lambda_a^m_n) (\lambda_a^j_i)$  (called a crossing relation)

$$V(r) = -\frac{4}{3} \cdot \frac{g^2}{4\pi} \cdot \frac{1}{r} \quad \text{for } (\bar{q}q)_{\underline{1}} \quad \text{and} \quad \frac{1}{6} \cdot \frac{g^2}{4\pi} \cdot \frac{1}{r} \quad \text{for } (\bar{q}q)_{\underline{8}}.$$

For the  $qq$  potential, an explicit decomposition leads us to

$$V(r) = -\frac{2}{3} \cdot \frac{g^2}{4\pi} \cdot \frac{1}{r} \quad \text{for } (qq)_{\underline{3}} \quad \text{and} \quad \frac{1}{3} \cdot \frac{g^2}{4\pi} \cdot \frac{1}{r} \quad \text{for } (qq)_{\underline{6}}.$$

The force responsible for binding is not entirely these coulombic forces, but their signs are reassuring, at least. We later introduce these vector bosons as the nonabelian gauge particles of color SU(3) and call the "gluons".